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Notes on $P(k)$-graphs and the conjecture of Kotzig

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Abstract

A $P(k)$-graph is a graph in which any two distinct vertices are joined by a single k-path. In this thesis we will discuss such graphs as well as the conjecture of Kotzig, both motivated by the study on edge extremal graphs without certain bipartite subgraphs. We will collect and prove some properties of $P(k)$-graphs and discuss attacks on the conjecture. As a certain approach we introduce the more general cycle-intersection-conjecture and prove some special cases. Further, some generalizations on above definitions are presented.

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1 Introduction

A central object of observation in extremal graph theory is the so called *Turán function* $\mathrm{ex}(n, F)$, where $n \in \mathbb{N}$ and F is a graph. It gives the maximal number of edges in an n-vertex graph without any subgraph isomorphic to F. Determining the exact value of this function for some fixed F but general $n \in \mathbb{N}$ seems to be hopeless in most cases, but the powerful theorem of Erdős and Stone (cf [6], p. 151) describes at least its asymptotic behavior very well. It states that for any graph F with chromatic number χ there holds

$$\mathrm{ex}(n, F) = \left(\frac{\chi - 2}{\chi - 1} + o(1)\right)\binom{n}{2}.$$

Unfortunately for bipartite F, for which $\chi = 2$, this only gives $\mathrm{ex}(n, F) = o(n^2)$. In particular, this includes the even cycles C_{2k}. Nearly fifty years ago (cf. [13]) Paul Erdős asked for a tighter determination of $\mathrm{ex}(n, C_{2k})$. Soon there was an answer for quadrangles C_4 (see [7]). It was found that the extremal edge count grows asymptotically like $\Theta(n^{3/2})$. Erdős himself conjectured $\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$. Today we know this is true for C_6 and C_{10}, but besides this there are no further final results (e.g. see [9]). The simple lower bound $\Omega(n^{1+\frac{1}{2k-1}})$ can be achieved by using Erdős' probabilistic method.

As all these results are aiming for asymptotic estimations, it is also an interesting task to explicitly construct C_{2k}-free graphs or to efficiently check whether a given graph is already edge extremal or not. Of course, a cycle C_{2k} is equivalent to two edge disjoint paths of length k between the same end vertices. So a cycle of length $2k$ can be avoided by demanding at most one path of length k between any pair of vertices. To maximize the number of edges one can get the idea that this can be achieved by additionally forcing at least one, hence exactly one path of length k between any two vertices. Graphs with this property are called *$P(k)$-graphs* and were first studied by Anton Kotzig in 1977 (see [11]). An also long known result is the classification of $P(2)$-graphs. This, again, is a result of Erdős together with Alfréd Rényi and Vera T. Sós and is known as the *Friendship Theorem* (see [7] or Section 1.3). It was proven that the edge count of $P(2)$-graphs is in $\Theta(n)$ and therefore does not provide infinitely many extremal graphs. For $k \geq 3$ it actually seems worse. Until today there is no $P(k)$-graph known for any $k \geq 3$ but there is also no proof that they do not exist. Their not-existence was conjectured by Kotzig after he proved this for $k \leq 7$ (see [12]). Some further solved cases for small k are listed in Section 1.4. Because of their questionable existence these graphs do not seem to fit the purpose of characterizing extremal C_{2k}-free graphs. One of the results of this thesis will be that even if there are such graphs, they at best generate a finite number of extremal graphs since their edge count is in $\mathcal{O}(n)$ (see Section 2.6).

This thesis will collect, prove and discuss properties of $P(k)$-graphs (see Chapter 2) as well as consider a certain attack on the conjecture of Kotzig (see Chapter 3). We also add some new results concerning this topic, especially about the degrees, symmetries and subgraphs of $P(k)$-graphs. Since we do not achieve a full proof, we hope that this collection may help a future researcher to get a quick overview on this topic, maybe useful for a final proof. Great work on $P(k)$-graphs and the conjecture was done by Kotzig himself (see [12]), by John A. Bondy in his survey on the topic (see [3]) and by Alexandr V. Kostochka (see [10]). Some generalized topics will be discussed in Chapter 4.

1.1 Definitions and notations

The definitions and notations of this thesis mainly will follow Diestel [6]. Variations from these standards are introduced below.

Throughout this text and if not mentioned otherwise let $G = (V, E)$ be a *finite graph* with vertex set V and edge set $E \subseteq \binom{V}{2}$. We will use $n := |V|$ as the number of vertices. The word *graph* refers to *simple graph* and so we will not discuss directed graphs, multi- or hypergraphs. Whenever we deal with another graph H, we use $V(H)$ to denote the vertex set and $E(H)$ to denote the edge set of H.

The set $\mathbb{N} = \{1, 2, 3, ...\}$ denotes the set of *natural numbers* greater than zero. If the zero should be included we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The length of a path P or cycle C will always be the number of its edges and is denoted by $|P|$ and $|C|$. Note that in the case of paths this does not equal the number of vertices which is denoted by $|V(P)|$. We use the abbreviation k-path and k-cycle for paths or cycles of length k. The same notation might be applied in other cases where the term of length is defined, e.g. for k-arcs, k-chords or k-loops (for the definitions of these terms see below or Chapter 3). A path intersecting a subgraph $H \subseteq G$ at most in its end vertices is called *H-crossing-free* (or simply crossing-free if the corresponding subgraph is clear). A path is said to *connect* its end vertices. If P is a path and $v, w \in V(P)$ two vertices, then vPw denotes the subpath of P between v and w. If P and Q are paths intersecting only in one end vertex $v \in V(P) \cap V(Q)$, then PQ means the concatenation of these paths at v. When it seems to be appropriate to emphasize the connection vertex, we also write PvQ. An analogous notation is used for arbitrary long path concatenations which also may close to a cycle.

We say two subgraphs $H_1, H_2 \subseteq G$ *intersect* in $v \in V$ if v is contained in both subgraphs. The vertex v is then called an *intersection vertex* or *crossing* of the two subgraphs H_1 and H_2.

A path that is a subgraph of a cycle C is called an *arc* of C. A path intersecting C exactly in its end vertices is called a *chord* of C, hence is C-crossing-free.

If $v, w \in V$ are two vertices and the shortest path between v and w is of length d, then d is called the *distance* of v and w and we denote it by $d(v, w)$. Of course $d(v, v) = 0$. We define

$$\begin{aligned} N_G^k(v) &:= \{w \in V \mid d(v, w) = k\}, \\ B_G^k(v) &:= \{w \in V \mid d(v, w) \leq k\}. \end{aligned}$$

to be the *k-neighborhood* of v or the *k-ball* around v. We will write $N_G(v)$ instead of $N_G^1(v)$ and $B_G(v)$ instead of $B_G^1(v)$. A vertex $w \in N_G(v)$ is called a *neighbor* of v. The *diameter* $\operatorname{diam}(G) := \max_{v,w \in V} d(v, w)$ is the largest distance of two vertices in G. The *eccentricity* of a vertex $v \in V$ is defined by $\epsilon(v) := \max_{w \in V} d(v, w)$. It is a measure for the distance of the farthest vertex from v. A vertex of minimal eccentricity is called a *center* of G. The eccentricity of a center is the *radius* $\operatorname{rad}(G)$ and can equivalently defined by

$$\operatorname{rad}(G) := \min_{v \in V} \max_{w \in V} d(v, w).$$

If a graph G is of radius one and v is a center of G then the graph is called *v-dominated*. Note that in this case all other vertices of G are adjacent to v.

The *degree* $d_G(v)$ of a vertex $v \in V$ is the number of vertices adjacent to v. It holds $d_G(V) = |N_G(v)|$. If $S \subseteq V$ is a set of vertices from G then $d_S(v)$ denotes the number of neighbors of v in S, hence $d_S(v) = |N_G(v) \cap S|$. The *minimal*, *maximal* and *average degree* of G are defined by

$$\delta_G := \min_{v \in V} d_G(v), \qquad \Delta_G := \max_{v \in V} d_G(v), \qquad \bar{d}_G := \frac{1}{n} \sum_{v \in V} d_G(v).$$

Most of the time the subscript G is dropped when the corresponding graph follows from the context.

We are going to use the Landau-notations $\mathcal{O}$ (big-O), o (small-O), Θ and Ω. Whenever this notation appears and if not mentioned otherwise, these asymptotics are used with respect to the vertex count n tending to positive infinity. As usual, we will write $f = \mathcal{O}(g)$ instead of $f \in \mathcal{O}(g)$. Accordingly for the other symbols.

We proceed with the definition of the central object of our observations:

Definition 1.1. A graph G on $n \geq 2$ vertices is called a *$P(k)$-graph* for some $k \in \mathbb{N}$ if any two distinct vertices $v, w \in V$ are connected by a unique k-path. The set of all $P(k)$-graphs will be denoted by $P(k)$.

The restriction to graphs on at least two vertices is useful because otherwise the empty graph and the graph on a single vertex would be $P(k)$-graphs for all $k \in \mathbb{N}$. We decided to exclude these trivial examples.

If there are two k-paths in G with the same end vertices we will call this a *double k-path*.

1.2 Basic observations on $P(k)$-graphs

Concerning $P(k)$-graphs there are some basic properties that do not need a long proof and we will collect them in this section.

Observation 1.2. All $P(k)$-graphs are connected and therefore each one contains an edge $e = \{v, w\}$. According to the definition there is a k-path between v and w and together with the edge e we got a cycle of length $k + 1$. If e would be contained in two $(k + 1)$-cycles then following the cycles from v to w (not over e) reveals two k-paths between these vertices. We conclude our most important observation: any edge is contained in *exactly one* $(k+1)$-cycle and hence G is uniquely decomposable into such cycles. Since adding a cycle raises the degree of all vertices by an even number we found that all degrees are even, thus G is Eulerian and for the minimal degree holds $\delta \geq 2$.

If $\{C_i\}$ is such an edge disjoint $(k+1)$-cycle-decomposition of G, then there are no other $(k+1)$-cycles besides these. If there would be another cycle, it would share an edge with an C_i. This is not possible as discussed in Observation 1.2. The number of $(k + 1)$-cycles will be denoted by c_{k+1}. Accordingly, the number of k-paths is p_k. Note that p_k is exactly known to be $\binom{n}{2}$ since there is exactly one path for any pair of vertices in G.

$P(k)$-graphs cannot be bipartite. A path P starting in a fixed vertex must change the partition class in each step. The partition class of the end vertex then only depends on the parity of the length of P. So all k-path starting in this vertex will end in the same partition class, hence never reach the other class. If the other class is not empty, not all vertices can be reached by such a path. In particular $\chi(G) \geq 3$ and G has to contain an odd cycle.

1.3 $P(2)$-graphs and the Friendship Theorem

The case $k = 2$ is one of the few cases in which we not only know that $P(k)$ is not empty but also know the full classification of contained graphs. Consider the following equivalent definition of $P(2)$-graphs:

> A graph G is called $P(2)$-graph if any two distinct vertices $v, w \in V$ share exactly one common neighbor, i.e. $N_G(v) \cap N_G(w)$ contains exactly one vertex.

The following class of graphs contains infinitely many $P(2)$-graphs.

Definition 1.3. A non-empty graph G is called a *windmill graph* (or just a *windmill*) if it satisfies one of the following equivalent definitions:

(i) G is the union of edge disjoint triangles, all overlapping in a unique *central vertex* v.

(ii) G is dominated by a *central vertex* v and $G - v$ is a 1-factor (i.e. G is 1-regular, hence a perfect matching).

Once we have chosen a central vertex the other vertices will be called *outer vertices*.

Proof of equivalence. (i) $\Rightarrow$ (ii): Choose v to be the vertex shared by all triangles. Since a triangle is dominated by all of its vertices, G is dominated by v. Triangles are complete graphs, so if two triangles share two or more vertices, they are sharing an edge. The triangles used in the definition are said to be edge disjoint and since they share already v, they are vertex disjoint with the only exception v. Removing v from G removes a single vertex from each triangle and leaves disjoint K_2's. This is a 1-factor.

(ii) $\Rightarrow$ (i): A 1-factor is the disjoint union of K_2's. By adding a vertex v and joining it to all the other vertices, each K_2 becomes a K_3. These triangles only share the vertex v. The triangles are pairwise edge disjoint, because sharing an edge means sharing at least two vertices. $\square$

Observation 1.4. We show the easy fact that these graphs belong to $P(2)$. Let $v \in V$ be a dominating vertex. Choose two vertices $a, b \in V$.

Case 1: The vertices a and b are outer vertices. Of course v is a common neighbor. If v' would be another common neighbor then v' would have degree 2 in $G - v$, which is a contradiction.

Case 2: One of the vertices is equal to v, say $a = v$. Then there is a unique neighbor b' of b in $G - v$. This vertex is the unique common neighbor of b and v.

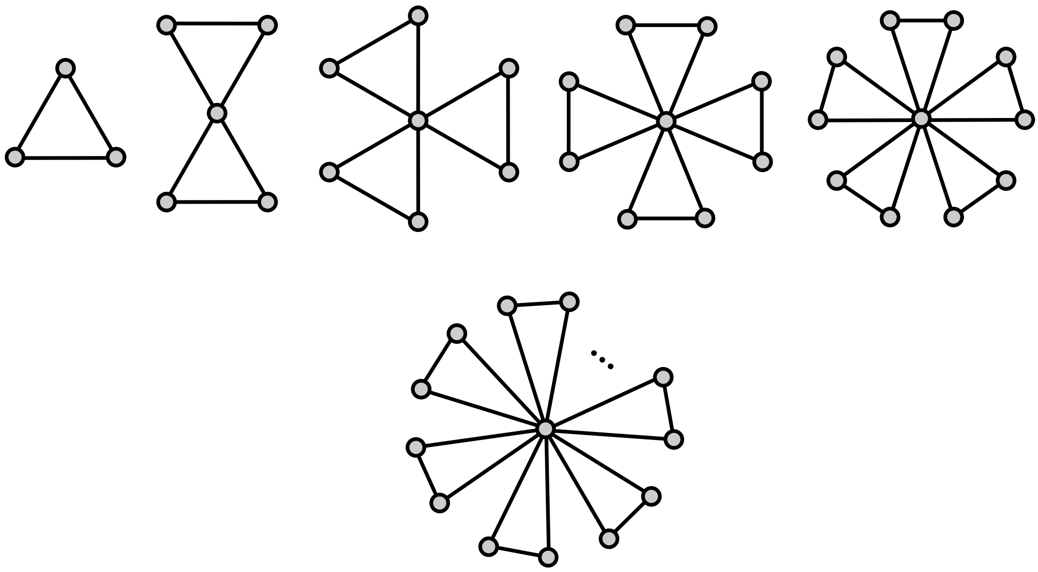

Figure 1.1: The general form of windmill graphs.

More interesting than that windmills are in $P(2)$ is the fact that these are the only $P(2)$-graphs at all. This is the non-trivial statement of the so called *Friendship Theorem*, first proved by Erdős, Rényi and Sós in [7].

Theorem 1.5 (The Friendship Theorem)**.** *The $P(2)$-graphs are exactly the windmills.*

The name of the theorem originates from the following way to express its statement:

> If there are $n \geq 2$ people and we know that any two of them have exactly one friend in common (and friendship is a symmetric relation), then there is a person which is friend with everyone (the *host* or *politician*). Moreover all other people occur in *couples*, i.e. they are friend with each other and the host, but with no one else. In particular n is odd.

Until today there is no easy and pure graph theoretical or combinatorial proof known. The original proof uses finite geometry in the final step. A much more elementary proof, the one we will present here, only uses simple linear algebra (cf. [2], p. 223 ff). The proof will basically go in two steps. First we show that a $P(2)$-graph which is not a windmill must be a regular graph. In the second step (the linear algebra step) we show that the triangle is the only regular $P(2)$-graph. To show the regularity we need two preparatory propositions.

Proposition 1.6. *$P(2)$-graphs are locally windmills, i.e. $G[B_G(v)]$ (the subgraph of G induced by v and its neighborhood) is a windmill for every vertex $v \in V$.*

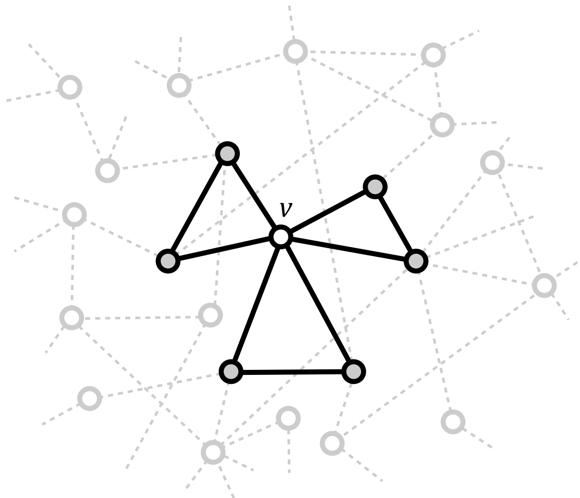

Figure 1.2: A locally induced windmill in a graph which itself is not a windmill. The neighborhood $N_G(v)$ (the gray filled vertices) induces a perfect matching.

Proof. Choose a vertex $v \in V$. Of course, $H := G[B_G(v)]$ is v-dominated. We show that removing v leaves a 1-factor and hence, by part (ii) of Definition 1.3, H is a windmill. Since $n \geq 2$ and G is connected we can choose a vertex $w \in H - v$. Since all vertices in $V(H - v)$ are neighbors of v, all neighbors of w in $V(H - v)$ are common neighbors of v and w. From the definition of $P(2)$-graphs follows that there is exactly one such shared neighbor. Therefore every vertex $w \in V(H - v)$ has exactly one neighbor in $V(H - v)$ and $V(H - v)$ induces a 1-factor. $\square$

We see that if a friendship graph G is of radius one then G is a windmill. But there is another sufficient condition for being a windmill.

Proposition 1.7. *If G is a $P(2)$-graph with minimal degree $\delta = 2$ then G is a windmill.*

Proof. Let $v \in V$ be a vertex of degree 2 and let w and w' its neighbors. Because G is locally a windmill (by Proposition 1.6) the vertices w and w' are adjacent. Define (see Figure 1.3)

$$A := N_G(w) \setminus \{v, w'\}, \qquad B := N_G(w') \setminus \{v, w\}.$$

If $u \in A \cap B$ then u and v are both common neighbors of w and w'. Therefore $A \cap B = \emptyset$. Now let $u \in A$ and $u' \in B$. If u and u' are adjacent, then u and w' are common neighbors of u' and w. So u has no neighbors in B and u' no neighbors in A. But nevertheless there must be a common neighbor s of u and u'. But this neighbor can neither lie in A or B nor can s be w, w' or v. So either $A = \emptyset$ or $B = \emptyset$, say B is

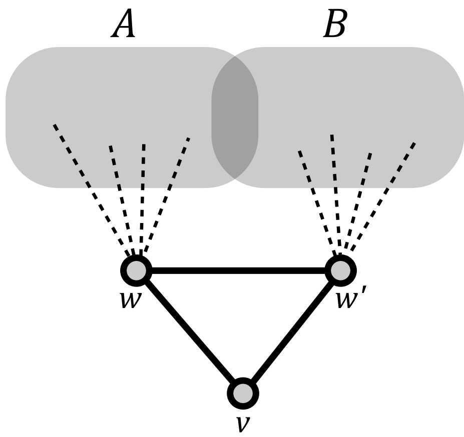

Figure 1.3: Configuration used in the proof of Proposition 1.7.

empty. Then w dominates G and as discussed above this already implies that G is a windmill. $\square$

A consequence of the last proposition in connection with Observation 1.2 is that for a friendship graph G that is not a windmill there must hold $\delta \geq 4$. We are now ready to proof the regularity statement as a further proposition:

Proposition 1.8. *If G is a $P(2)$-graph but no windmill, then G is d-regular with $d \geq 4$.*

Proof. We give a proof in two steps. For the first step we show that any two vertices at distance two are of the same degree. Assuming that G is no windmill we can show in step two that this suffices to imply the regularity of G.

Step 1: Let $a, b \in V$ two vertices of distance two and v their common neighbor. Further let a' be the common neighbor of v and a, and b' the common neighbor of v and b. Define

$$A := N_G(a) \setminus \{a', v\}, \qquad B := N_G(b) \setminus \{b', v\}$$

and observe that this gives a configuration as viewed in Figure 1.4. Choose an arbitrary vertex $a'' \in A$. We show that the unique common neighbor b'' of a'' and b is in B. It holds

- $b'' \neq v$ because a' is already a common neighbor of a and v.
- $b'' \neq b'$ because v is already a common neighbor of a and b'.

All other neighbors of b are in B and so $b'' \in B$. This neighbor of a'' in B is unique because all neighbors of a'' in B share the neighbors a'' amd b. So every $a'' \in A$ has exactly one neighbor in B and we found $|A| \geq |B|$. Because of the obvious symmetry

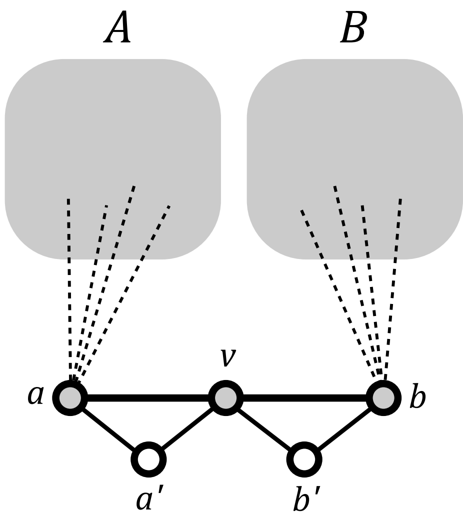

Figure 1.4: Configuration used in the proof of Proposition 1.8.

we get $|A| = |B|$ and because $d_G(a) = |A| + 2$ and $d_G(b) = |B| + 2$ we also get $d_G(a) = d_G(b)$.

Step 2: If there are no vertices of distance two then the graph is complete, hence regular. So assume $a, b \in V$ of distance two. We know that $d_G(a) = d_G(b)$. Except for the common neighbor v of a and b all other vertices are either of distance two to a or of distance two to b and so all vertices in $V \setminus \{v\}$ are of the same degree. Since G is no windmill, G is not dominated by v and therefore there must be a vertex $u \in V$ of distance two from v and we are done. □

The second part of the proof of the Friendship Theorem is the one using techniques from linear algebra. The connection to this part of mathematics arises from the possibility to represent a graph in the form of a matrix, here we use the adjacency matrix. In the final step of the proof we will need the following observation:

Observation 1.9. Consider the set of possible paths of length two starting in an arbitrary vertex $v \in V$. For the first step of the path there are d possibilities. For the second step there are only $d-1$ because we cannot go back to v. So there are $d(d-1)$ such paths. Since we can reach any other vertex from v by exactly one path of length two, there must hold $d(d-1) = n-1$ and this rearranges to

$$n = d^2 - d + 1. \tag{1.1}$$

Proof of Theorem 1.5. A 1-regular graph cannot be a $P(2)$-graph because it would consist of a single edge. Let G be a regular $P(2)$-graph of degree $d \geq 2$ and A its adjacency matrix. We recall that the entry in row i and column j of A^2 gives the number of paths of length two from vertex i to vertex j. So

$$A^2 = \begin{pmatrix} d & 1 & 1 & \cdots & 1 \\ 1 & d & 1 & \cdots & 1 \\ 1 & 1 & d & & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & d \end{pmatrix} = \mathbb{E} + (d-1)\mathbb{I},$$

where $\mathbb{E}$ is the matrix only filled with ones and $\mathbb{I}$ is the identity matrix. $\mathbb{E}$ has the simple eigenvalue n and the multiple eigenvalue 0 of multiplicity $n-1$. Since A^2 is a polynomial in $\mathbb{E}$ its eigenvalues are

$$\lambda_1 = n + d - 1, \qquad \lambda_2 = d - 1,$$

also with the multiplicities 1 and $n-1$. We know that the eigenvalues of A are of the form $\pm\sqrt{\lambda_1}$ and $\pm\sqrt{\lambda_2}$ and since the trace of A is zero, the sum of the eigenvalues has to vanish. Assume w.l.o.g. that A has the eigenvalue $\sqrt{\lambda_1}$ and let c be the multiplicity of $\sqrt{\lambda_2}$. There must hold

$$\sqrt{\lambda_1} + c\sqrt{\lambda_2} - (n-1-c)\sqrt{\lambda_2} = 0.$$

By rearranging the terms we get

$$\sqrt{\frac{\lambda_1}{\lambda_2}} = n - 1 - 2c$$

and because the right hand side is an integer also the left hand side has to be an integer. So there must be an $f \in \mathbb{N}_0$ with $\lambda_1 = \lambda_2 f^2$. Note that $n = \lambda_1 - \lambda_2$ and $d = \lambda_2 + 1$. By substituting this into Eq. (1.1) from Observation 1.9 we get $\lambda_1 = \lambda_2^2 + 2\lambda_2 + 1$. Finally we derive

$$f^2 = \lambda_2 + 2 + \frac{1}{\lambda_2}$$

and here f^2 can only be an integer if $\lambda_2 = 1$ which means $d = 2$. So G is a cycle, but the triangle is the only cycle in $P(2)$. Since the K_3 is a windmill we are done. □

1.4 The conjecture of Kotzig

In 1977 Anton Kotzig stated the following conjecture:

Conjecture 1.10 (Kotzig). *There are no $P(k)$-graphs for any $k \geq 3$.*

Kotzig himself proved it up to $k = 8$ by using statements on even cycles (see Section 2.1). Kostochka was able to show that a longest cycle C in G satisfies $k+5 \leq |C| \leq \frac{4}{3}k-2$ (see [10]). To make this possible there must hold $k+5 \leq \frac{4}{3}k-2$ which rearranges to $k \geq 21$. This proves the conjecture for the cases up to $k = 20$. In a concluding remark it is asserted that this bound can be extended to treat all cases $k \leq 30$. In a paper of K. Xing and B. Hu [16] the authors claimed to have proven the cases $k \geq 11$ and the problem seemed solved. But in the year 2000 R. Häggkvist pointed out that the work was based on the unproven assumption that there always exists a $(2k-8)$-cycle in a $P(k)$-graph. In conclusion, this paper only shows that such cycles are forbidden. Because of the hard work on the conjecture without a final result, in [2] a proof is indicated as "currently out of reach".

Example 1.11. We show that conjecture Conjecture 1.10 holds for $k = 3$. According to Observation 1.2 there is a 4-cycle C in G. Observe that neither C, C with a single chord nor K_4 are $P(3)$-graphs. Consequently there has to be a vertex $v \notin V(C)$ but adjacent to C. Assume v is a neighbor of $w \in V(C)$. The cycle C can be divided into two 2-arcs with end vertex w. Let w' be the other common end vertex of these arcs. Now, each one of these arcs appended to the edge $\{v, w\}$ gives a 3-path between v and w'. This contradicts the uniqueness of 3-paths and hence there are no $P(3)$-graphs.

A similar argument works for $k = 5$. More difficult are the cases with even k. E.g. for $k = 4$ we need to discuss several different cases. All $k \leq 5$ are discussed and solved later by using the *cycle-intersection-conjecture* in Chapter 3.

2 Properties of $P(k)$-graphs

2.1 Cycles in $P(k)$-graphs

Whenever there are two k-paths between two vertices $v, w \in V$ this gives rise to at least one cycle. So, to avoid such multiple paths we should keep a closer eye on the cycle structure of our graphs. Especially the even cycles seem to be dangerous for the $P(k)$-property as seen in Example 1.11 and more generally explained in the following observation.

Observation 2.1. The most important thing about even cycles for this topic is that there are, in contrast to odd cycles, so called *antipodal vertices*. If C is an even cycle of length 2ℓ and $v \in V(C)$, there is a unique vertex $v' \in V(C)$ (the antipodal vertex) that together with v divides C into two arcs of length ℓ. These arcs are two crossing-free ℓ-paths between the same vertices. Thus, if $\ell \leq k$ and in any vertex $v \in V(C)$ there starts a path of length $k - \ell$ that does not intersect C again, we already found a double k-path as seen in Fig. 2.1. For the same reason a chord of C must have length at most $k - \ell$ because a longer chord would contain such an undesirable $(k - \ell)$-path. In Lemma 2.5 we will push this further. We summarize: even cycles of length 2ℓ with $\ell \in \{2, ..., k\}$ are potentially dangerous (where actually $\ell = k$ is trivially dangerous and forbidden).

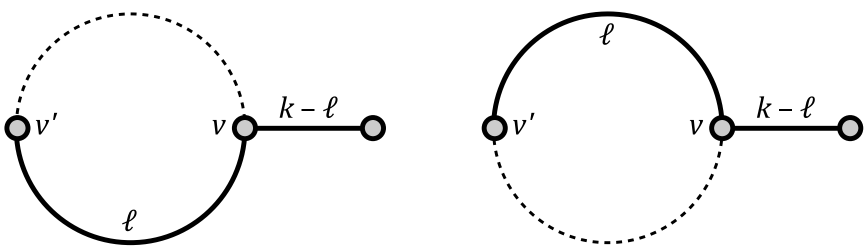

Figure 2.1: The two possible k-paths that may emerge from the existence of an even 2ℓ-cycle.

This, in general, makes the existence of $P(k)$-graphs more unlikely for odd k because these graphs necessarily contain even $(k + 1)$-cycles (see Observation 2.1). A main objective of this section will be to show that this is not just a problem for odd k. The following weak criterion suffices to imply an even cycle.

Lemma 2.2. *Let $v, w \in V$ be two vertices having three pairwise crossing-free paths between them. Then G contains an even cycle.*

Proof. Call the paths $P_i, i = 1, 2, 3$. The union of two of these paths is a cycle with the length the sum of the lengths of the building paths. If $|P_1| + |P_2|$ is even then the union $P_1 \cup P_2$ is an even cycle and we are done. If the sum is odd this means $|P_1|$ and $|P_2|$ are of different parity. So one of them must have the same parity as $|P_3|$, say P_1. Hence $P_1 \cup P_3$ is an even cycle. □

We will show that the configuration of three crossing-free paths is indeed present. We need a preparatory lemma.

Lemma 2.3. *Every cycle in a $P(k)$-graph with $k \geq 3$ has a chord of length $\ell \in \{1, ..., k\}$.*

Proof. Call the cycle C. Choose two vertices $v, w \in V(C)$ which are not connected by a k-path in C. Observe that this is always possible if $k \geq 3$. So the k-path P from v to w has to have edges outside of C. If P and C intersect in exactly s vertices, this divides P into $s+1$ C-crossing-free subpaths and at least one of them, call it P', has to contain edges outside of C. Of course P' is a chord of C and as a subpath of P it must be of length at most k. □

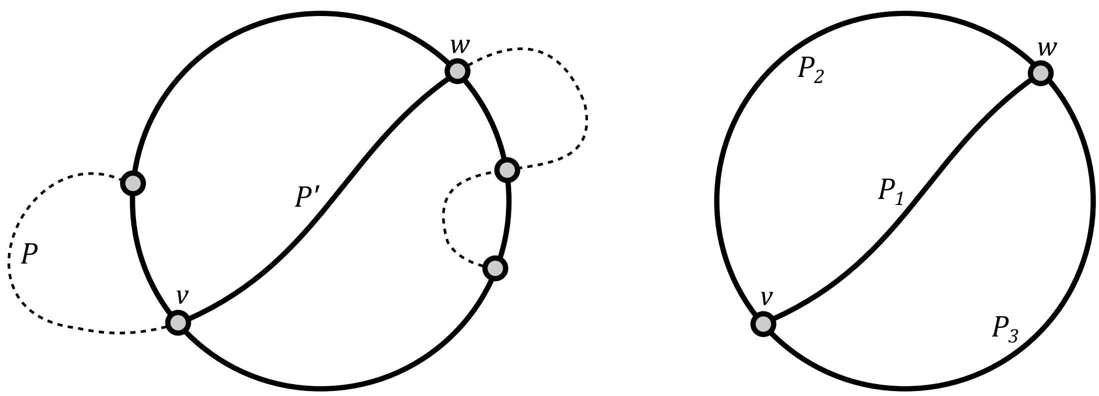

Figure 2.2: The left side displays the configuration used in the proof of Lemma 2.3. The right side shows the configuration used in the proof of Lemma 2.4. At least two of the three paths P_1, P_2 and P_3 combine to an even cycle.

Lemma 2.4. *For $k \geq 3$ every $P(k)$-graph contains an even cycle of length 2ℓ for an $\ell \in \{2, ..., k-1\}$.*

Proof. This is clear for odd k (by Observation 2.1), because the $(k+1)$-cycles are even. For even k choose an odd $(k+1)$-cycle C. According to Lemma 2.3 C has a chord P_1 between two vertices $v, w \in V(C)$. The vertices v and w divide C into two arcs P_2 and P_3. By Lemma 2.2, the three pairwise crossing-free paths P_1, P_2 and P_3 between v and w imply an even cycle. Since the length of the paths are in $\{1, ..., k\}$ and $2k$-cycles and multi-edges are forbidden, we obtain an even cycle of length 2ℓ with $\ell \in \{2, ..., k-1\}$. □

Actually we can say more about the possible cycle lengths. Kotzig himself proved that there cannot exist cycles of length $\ell \in \{4, 2k-6, 2k-4, 2k-2\}$. Further, in [16] it was shown that the existence of $(2k-8)$-cycles leads to a contradiction. The absence of quadrangles can be explained by observing that any possible path of length k from a vertex in a C_4 to its antipodal vertex induces a double k-path as illustrated in Fig. 2.3. This restricts the length 2ℓ of an even cycle to $6 \leq 2\ell \leq 2k-10$. Solving for k gives $k \geq 8$. In the case $k = 9$ there is a 10-cycle, but here this is also a $(2k-8)$-cycle which is forbidden. An even stronger statement by Kostochka (see [10]) proves the non-existence of cycles longer than $\frac{4}{3}k-2$. He also proved the existence of a cycle of length at least $k+5$. We will give a proof of the non-existence of $(2k-2)$-cycles by the following more general lemma. From Observation 2.1 we know that a chord of an even 2ℓ-cycle is at most of length $k-\ell$. We can tighten this bound:

Lemma 2.5. *Let G be a $P(k)$-graph and C an even cycle of length 2ℓ, $\ell \in \{2, ..., k-1\}$ in G. All chords of C are shorter than $k-\ell$.*

Proof. Assume P to be a chord of C and let $v, w \in V(C)$ be its end vertices. If P is of length at least $k-\ell+1$ we already found a double k-path in Observation 2.1. So assume $|P| = k-\ell$. We construct a double k-path as seen in Fig. 2.4. The vertices v and w divide C into two arcs Q and R, say $|Q| \geq |R|$. Choose an inner vertex u from R. This again divides R into two subpaths R_1 and R_2. Also divide Q into the subpaths Q_1 and Q_2 of lengths

$$|Q_1| = \frac{1}{2}(|Q| + |R_1| - |R_2|), \qquad |Q_2| = \frac{1}{2}(|Q| - |R_1| + |R_2|).$$

Observe that these lengths are positive integers by using that $|R|$ and $|Q|$ are of the same parity. Further it holds $|Q| = |Q_1| + |Q_2|$, so these arcs decompose Q. Let u' be the common end vertex of Q_1 and Q_2. Also choose the division of Q as shown in Fig. 2.4. Now we found the paths

$$S_1 := uR_2wPvQ_1u', \qquad S_2 := uR_1vPwQ_2u'$$

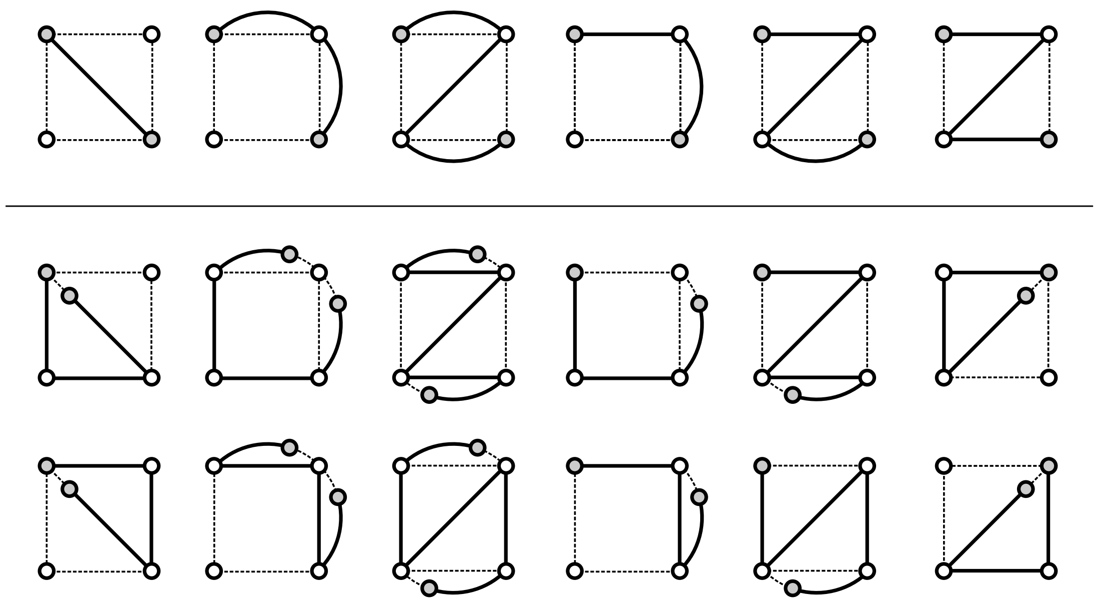

Figure 2.3: The upper row shows the possible ways to connect two antipodal vertices of a C_4 (the gray vertices) by a k-path. The two lower rows then show two k-paths induced by the configuration above. These two paths may use the two 2-arcs of the C_4 as well as a chord between antipodal vertices, if possible. The gray colored vertices in the lower rows are the end vertices of the two k-paths.

to be of the same length k. We calculate the length exemplary for S_1:

$$\begin{aligned}|S_1| &= |R_2| + |P| + |Q_1| \\ &= |R_2| + |P| + \frac{1}{2}(|Q| + |R_1| - |R_2|) \\ &= |P| + \frac{1}{2}(|Q| + |R_1| + |R_2|) \\ &= |P| + \ell = k.\end{aligned}$$

S_1 and S_2 share the end vertices u and u' and so we found a double k-path. This cannot be, hence the chord must have length smaller than $k - \ell$. $\square$

Corollary 2.6. *A $P(k)$-graph does not contain a $(2k-2)$-cycle.*

Proof. According to Lemma 2.3 such a cycle must have a chord. But because of Lemma 2.5 this chord would be of length zero which is not possible. $\square$

We summarize the results on cycles obtained above and by considerations in [10], [11] and [16]

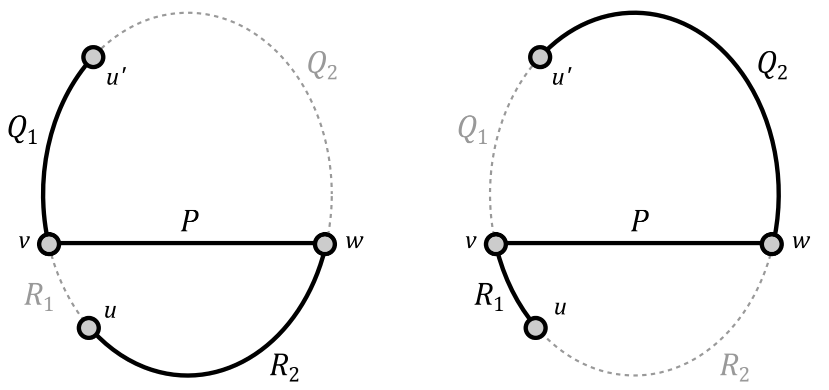

Figure 2.4: Configuration used in the proof of Lemma 2.5. A $(k-\ell)$-chord P induces a double k-path as shown.

Lemma 2.7. *Let G be a $P(k)$-graph for some $k \geq 3$. The following statements hold:*

(i) G contains an even cycle of length 2ℓ for an $\ell \in \{3, ..., k-5\}$ (cf. [11], [16]).

(ii) G does not contain an even 2ℓ-cycle with $\ell \in \{2, k-4, k-3, k-2, k-1, k\}$ (cf. [11], [16]).

(iii) G contains a cycle of length at least $k+5$ (cf. [10]).

(iv) G does not contain cycles longer than $\frac{4}{3}k - 2$ (cf. [10]).

When we indicated even cycles as potentially dangerous to induce double k-paths we only spoke about even cycles of lengths at most $2k$. Item (iv) now shows that this is nothing to care about: Whenever there is an even cycle we do not have to check the length to know of the problems arising.

2.2 Dependent substructures

In the last section we already discussed the problems arising with even cycles. We mentioned that a path starting in an even cycle and not intersecting it again can cause a double k-path if it is long enough. So we learned that the path structure that is left when removing an even cycle cannot be very rich. In detail, we will see that cycles and paths of certain lengths are always strongly entangled. An additional result will be an upper bound for the length of a longest path and cycle in a $P(k)$-graph.

Lemma 2.8. *Let G be a $P(k)$-graph and C an even 2ℓ-cycle in G. It holds:*

(i) All cycles of length at least $k - \ell$ intersect C.

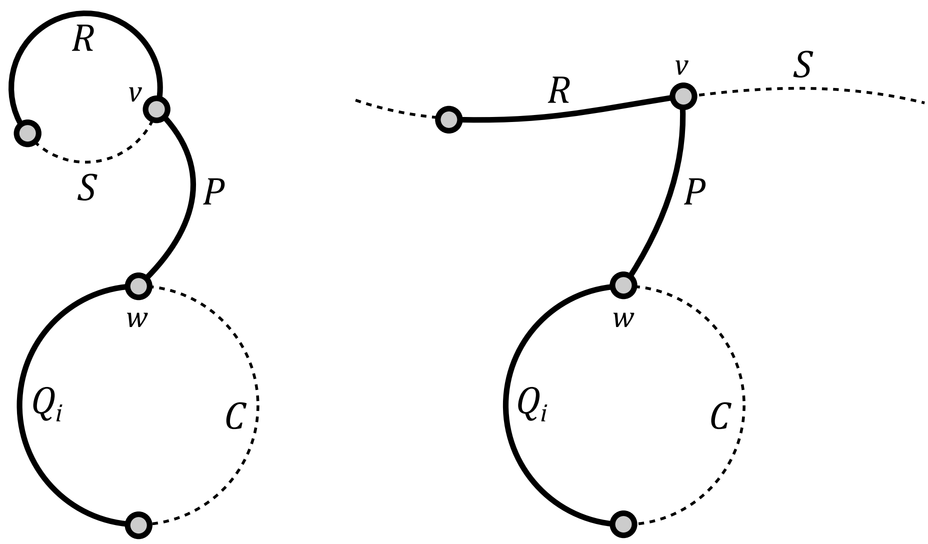

Figure 2.5: Configuration used in the proof of Lemma 2.8.

(ii) All paths of length at least $2(k-\ell)-3$ *intersect* C.

Proof. The idea is to assume the converse and then construct a double k-path as illustrated in Figure 2.5. We treat part (i) and (ii) together by letting S either a cycle of length at least $k-\ell$ or a path of length at least $2(k-\ell)-3$. Note that $\ell < k$ by Lemma 2.7. Assume S to not intersect C. Since G is connected we can find a path P from S to C where only the end vertices are contained in S or C. We call these end vertices $v \in V(S)$ and $w \in V(C)$. Observe that it is always possible to choose a path R of length $k-\ell-|P| < k-\ell$ in S starting in v. But then joining P and R in v gives rise to a $(k-\ell)$-path that starts in w but not intersects C again. This is a contradiction according to Observation 2.1. So C and S have to intersect. $\square$

If we already know that a path or cycle intersects a certain even cycle we can go further and estimate the number of intersections.

Lemma 2.9. *Let G be a $P(k)$-graph and C an even cycle in G of length 2ℓ.*

(i) A cycle of length $\ell' \geq k-\ell$ intersects C in at least $\frac{\ell'}{k-\ell}$ vertices.

(ii) A path of length $\ell' \geq 2(k-\ell)-3$ intersects C in at least $\frac{\ell'+2}{k-\ell}-1$ vertices.

Proof. Let S be a cycle or path of the required length which therefore intersects C in $s \geq 1$ vertices (see Lemma 2.8).

(i): We remove the intersection vertices from S which therefore decays into at most s subpaths. Since S has ℓ' vertices there are $\ell'-s$ vertices left for the subpaths

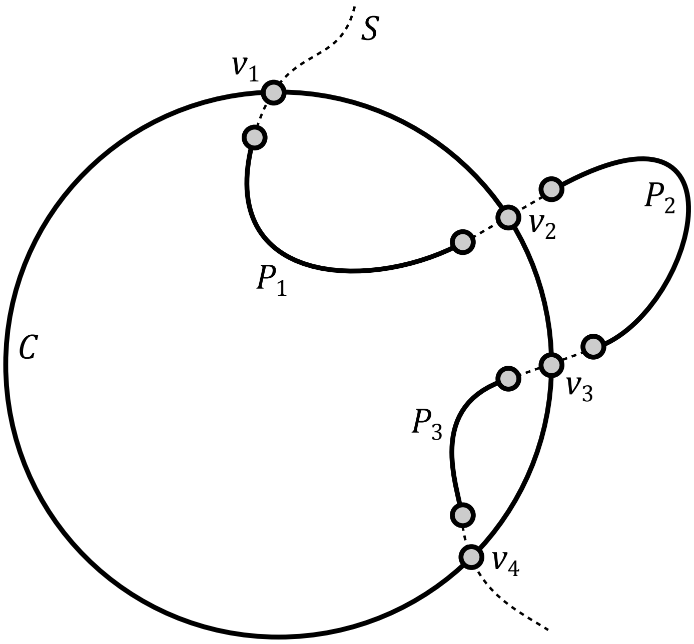

Figure 2.6: Construction used in the proof of Lemma 2.9. The path (or cycle) S is divided into subpaths P_i by removing its crossings $v_1, ..., v_s \in V(S)$ with C. If $|P_i| \geq k - \ell - 1$ this leads to a double k-path according to Observation 2.1.

and hence one of them, call it P, must consist of at least $\frac{\ell' - s}{s}$ vertices and has length at least $\frac{\ell' - s}{s} - 1$. When re-adding the intersection vertices an end vertex of P is now joined to the cycle C, thus defines a path of length $|P| + 1$ starting in C and not intersecting it again. From Observation 2.1 we know that such a path must have length at most $k - \ell - 1$. We derive

$$k - \ell - 1 \geq |P| + 1 \geq \frac{\ell' - s}{s} \quad \Rightarrow \quad s \geq \frac{\ell'}{k - \ell}.$$

(ii): The proof is analog to (i) but removing s intersection vertices divides S into at most $s + 1$ subpaths. Further S itself has $\ell' + 1$ vertices, so removing the crossings leaves a subpath of length at least $\frac{\ell' + 1 - s}{s + 1} - 1$. The same considerations as in (i) lead to the lower bound for s. □

The result of the last lemma can be slightly enhanced when requiring $s \geq 2$. The crossings of S and C divide S into subpaths and most of these are chords of C. Using Lemma 2.5 and a more detailed investigation can yield tighter lower bounds on s but the ones derived above are sufficient for the further approach.

Observation 2.10. What should we keep in mind from Lemma 2.8 and 2.9? We list some of the most useful consequences for the further approach:

1. If k is odd, then all $(k+1)$-cycles intersect each other at least twice. This follows directly from Lemma 2.9 by setting $2\ell := k+1$ and $\ell' := k+1$. In fact, for general k two 2ℓ-cycles already intersect each other if $\ell \geq \frac{1}{3}k$, and if we demand $\ell \geq \frac{1}{2}k$ they even intersect twice.

2. Cycles of length at least $k-1$ intersect all even cycles at least twice, e.g. $(k+1)$-cycles are such cycles. A $(k-2)$-cycle is intersected by all even cycles at least once.

3. Paths of length at least $2k-7$ intersect all even cycles.

As an additional result we can bound the length of general cycles and paths in $P(k)$-graphs. Surprisingly this upper bound only depends on k. The bound attained for cycles is weaker than the one achieved by Kostochka in [10].

Corollary 2.11. *Let G be a $P(k)$-graph for $k \geq 3$ and C an even cycle of length 2ℓ. Let C' be a cycle and P' be path in G. It holds $|C'| \leq 2\ell(k-\ell)$ and $|P'| \leq (2\ell+1)(k-\ell)-2$. In particular:*

$$|C'| \leq \frac{1}{2}k^2, \qquad |P'| \leq \frac{1}{2}k(k+1) - \frac{15}{8}. \tag{2.1}$$

Proof. Assume $|C'| > 2\ell(k-\ell)$ or $|P'| > (2\ell+1)(k-\ell)-2$. Note that this is always long enough to enforce a crossing with C by Lemma 2.8. Now, when using Lemma 2.9 to estimate the number s of these crossings with C, we obtain $s > 2\ell$. This, of course, is not possible since $|C| = 2\ell$. The inequalities (2.1) can be achieved by observing that the upper bound is maximized for $\ell = \frac{1}{2}k$ for cycles and $\ell = \frac{1}{4}(2k-1)$ for paths. □

A powerful result of this section is that the cycle structure of a $P(k)$-graph for odd k is pretty entangled in the sense that all $(k+1)$-cycles intersect each other at least twice. This will be the basis of the attack presented in Chapter 3. To max out the applicability of this approach, we show that this is not a peculiarity for odd k.

Lemma 2.12. *Let G be a $P(k)$-graph for $k \geq 3$. Any two $(k+1)$-cycles in G intersect at least twice.*

Proof. This is clear for odd k as discussed in Observation 2.10. So let k be even and C and C' two odd $(k+1)$-cycles in G. From Lemma 2.4 we know that C has a chord P that together with some arc P' of C builds an even cycle $\tilde{C}$ of length 2ℓ. Say the end vertices of P are $v, w \in V(C)$. In Observation 2.10 we mentioned that C' has

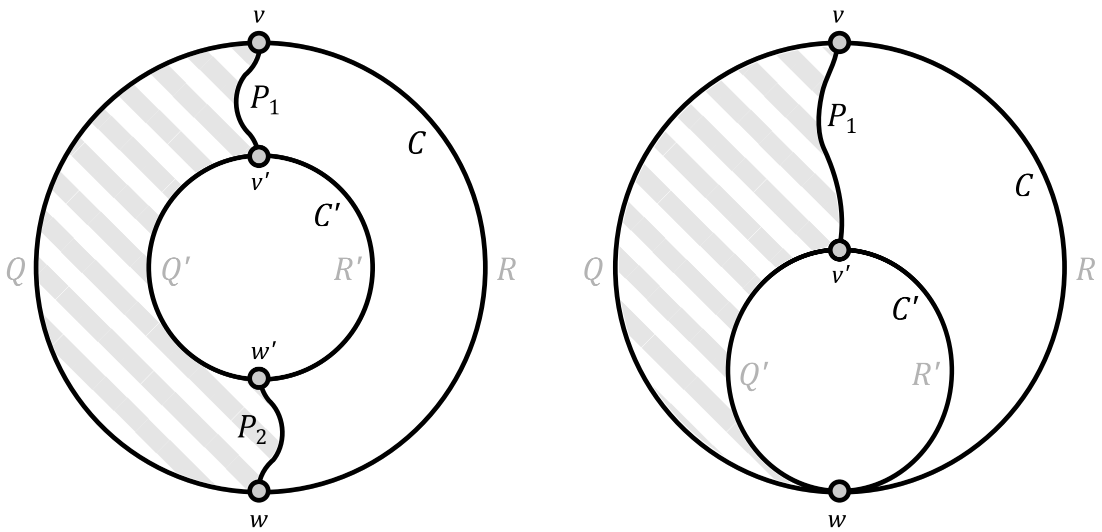

Figure 2.7: Construction used in the proof of Lemma 2.12. The configuration on the right hand side can be seen as a special case of the configuration on the left hand side when $|P_2| = 0$ and $w = w'$. The shaded area represents the interior of the even cycle $S := QP_1Q'P_2$ used in the proof.

to intersect all even cycles at least twice, especially C' intersects $\tilde{C}$ twice. If two of the crossings are in P' we found two intersections of C and C' and so we are done. Hence, consider the following two cases:

(i) If only one crossing of C' and $\tilde{C}$ lies in P', then there has to be another one in P. Choose the first intersection vertex v' of P and C' by starting at v and following P. Let $P_1 := vPv'$ be the subpath of P between v and v'.

(ii) If there is no crossing of C' and P', choose the first and last intersection vertex of C' and P (again with respect to the order induced by P, starting in v). Call these vertices v' and w'. Also define the subpaths $P_1 := vPv'$ and $P_2 := wPw'$.

Note that this gives rise to a configuration as seen in Fig. 2.7 and further that the first case can be seen as the second one by setting $|P_2| = 0$ and defining $w = w'$ to be the only intersection of C and C'. So we can treat the cases together. The vertices v and w divide C into two arcs Q and R. Respectively C' is divided into the arcs Q' and R' by v' and w'. Observe that either $QP_1Q'P_2$ or $RP_1Q'P_2$ is an even cycle because of Lemma 2.2 and C is odd. W.l.o.g. assume the former one, call it S, to be even. The paths R and R' are chords of S and according to Lemma 2.5 we know there must hold

$$\frac{1}{2}|S| + |R| < k, \qquad \frac{1}{2}|S| + |R'| < k.$$

We consider the sum of these inequalities and derive

$$\begin{aligned} 2k &> |S| + |R| + |R'| \\ &= |P_1| + |P_2| + |Q| + |Q'| + |R| + |R'| \\ &= |P_1| + |P_2| + 2(k+1) \end{aligned}$$

which rearranges to $|P_1| + |P_2| + 1 < 0$. This, of course, is a contradiction and we conclude that C and C' have to intersect. $\square$

In Chapter 3 we will extend the result above: two $(k+1)$-cycles have to intersect at least five but at most $k-1$ times. Since this implies $5 \leq k-1$, this will prove the conjecture for $k \leq 5$.

2.3 Connectivity

Lemma 2.13. *For $k \geq 3$ every $P(k)$-graph is 2-connected.*

Proof. We know G is connected. Consider the blocks (2-connected components) of G. A block always contains an edge. If e is an edge and C the $(k+1)$-cycle e belongs to, then e and C are part of the same block. Hence each block contains a $(k+1)$-cycle. But two distinct blocks intersect at most in a single node (a cut point). For $k \geq 3$ this is a contradiction to Lemma 2.12 which states that two $(k+1)$-cycles have to intersect at least twice. Thus, G consists of a single block and is 2-connected.

$\square$

Besides the vertex connectivity κ we can discuss the edge connectivity κ' which will turn out to be always even. Actually we can see more.

Observation 2.14. When partitioning the set V of vertices into two disjoint sets $V_1, V_2 \subseteq V$ with $V_1 \cup V_2 = V$, the set M of edges passing between vertices from different sets is called a *cut* of G. Each cycle in G must pass the cut an even number of times because for each transition from V_1 to V_2 it has to return to V_1 over another unused edge. Further, by Observation 1.2 G is decomposable into $(k+1)$-cycles C_i. This induces a decomposition of M into the sets $M_i := M \cap E(C_i)$. Since all the sets M_i are pairwise disjoint and of even size, also $|M|$ has to be even.

We recall that κ' is the size of the smallest cut, hence even by the observation above.

2.4 Subgraphs of $P(k)$-graphs

To prove properties for all $P(k)$-graphs it would be helpful to use induction on a simple graph parameter, e.g. the number of vertices or edges. We observe that this is not so easy:

Observation 2.15. Every edge $e \in E$ is contained in a $(k+1)$-cycle. Such a cycle contains $k+1$ paths of length k and only one of them does not contain e. So every edge is contained in at least k paths of length k. This means that removing a single edge will not result in a $P(k)$-graph anymore. The same holds when removing vertices. By the same considerations as above we can conclude that a vertex is always an inner vertex (a non-end vertex) of a k-path. This shows that a simple induction on the number of vertices or edges is not possible.

Lemma 2.16. *Let G be a $P(k)$-graph for some $k \geq 3$. A proper subgraph of G cannot be a $P(k)$-graph on its own.*

Proof. Assume $H \subset G$ to be a proper subgraph of G and also a $P(k)$-graph. From Observation 2.15 we know that only removing edges will not result in a $P(k)$-graph. So $G - H$ is not empty. Let H' be a single connected component of $G - H$. Further define ∂H to be the set of vertices in H with neighbors in H' and respectively $\partial H'$ the set of vertices in H' with neighbors in H. The proof will proceed in two steps. Because all pairs of distinct vertices from H do already have a connecting k-path in H, we have to ensure that including H' does not give rise to another possible k-path between the same end vertices. In step 1 we will derive that because of this, the vertices of $\partial H'$ have to be of a certain large distance in H'. In the second step this will cause a contradiction when considering $(k+1)$-cycles containing edges from H'.

Step 1: The set ∂H must be of size of at least two because it is a separator of G and according to Lemma 2.13 G is 2-connected for $k \geq 3$. Choose $v, w \in \partial H$ and for each one a neighbor in H', say v' and w' (see Fig. 2.8). Since H' is connected there is a path $P \subseteq H'$ between v' and w' which is possibly of length zero if $v' = w'$. Further we have a k-path $Q \subseteq H$ between v and w. This forms a cycle $C := vv'Pw'wQv$ of length $|P| + k + 2$. Choose a subpath Q' of Q of length $|P| + 2$. This is possible if $|P| \leq k-2$. Q' is an arc of C. The other arc is of length $(|P|+k+2)-(|P|+2) = k$, contains vertices in H' and connects two vertices in H. But because H is already a $P(k)$-graph there is a k-path with the same end vertices completely contained in H. This is a contradiction. We conclude $|P| \geq k-1$, in particular $v' \neq w'$ and $|V(H')| \geq |P| + 1 \geq k$.

Step 2: Because $|V(H')| \geq k$ (from step 1) and H' was chosen to be connected, we can choose an edge e in H' and a $(k+1)$-cycle $C \subseteq H$. From Lemma 2.12 we know that the $(k+1)$-cycle C' that contains e must intersect C at least twice, thus must contain two vertices from H. Also C' must have an arc in H' of length at least $k-1$ between the vertices connected to H (by step 1). So C' contains at least $k+2$ vertices. But this is a contradiction and we are done. $\square$

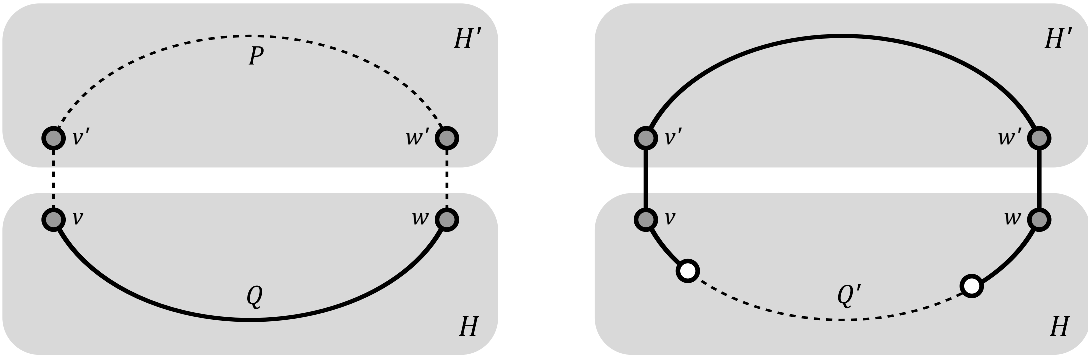

Figure 2.8: Configuration used in the first step of the proof of Lemma 2.16. The right hand side emphasizes the complementary arc to Q' in C.

Note that the assumption $k \geq 3$ is essential for the lemma because a triangle is a $P(2)$-graph but subgraph of all windmills.

2.5 Symmetries

As we will see in this section, $P(k)$-graphs are either completely asymmetric or of a certain rotational symmetry.

Lemma 2.17. *An automorphism $f \in \mathrm{Aut}(G)$ of a $P(k)$-graph G is either the identity or has at most one fixed point, i.e. at most one vertex $v \in V$ with $f(v) = v$.*

Proof. Let $H \subseteq G$ the subgraph of G induced by the set of fixed points of f. If $f = \mathrm{id}$ then $G = H$. Assume H to be a proper subgraph with $|V(H)| \geq 2$. According to Lemma 2.16 H is no $P(k)$-graph on its own. Hence, there are two vertices $v, w \in V(H)$ with the property that the connecting k-path $P = vv_1 \cdots v_{k-1}w$ (in G) is not completely contained in H. But then the image under f is $P' = vf(v_1) \cdots f(v_{k-1})w$ which is different from P because some of the vertices $v_1, ..., v_{k-1}$ are no fixed points of f. So we found two k-paths P and P' between v and w and this is a contradiction. Hence $|V(H)| \leq 1$. $\square$

This result suffices to classify all possible symmetries of $P(k)$-graphs.

Corollary 2.18. *For $k \geq 3$ every automorphism of a $P(k)$-graph is either the identity or a rotation, i.e. there is at most one fixed point and the orbits of all other vertices are equally long.*

Proof. From Lemma 2.17 we know that if $f \in \mathrm{Aut}(G)$ is not the identity then there is at most one fixed point. Now assume there are two vertices $v, w \in V(G)$ which are no fixed points but have orbits of different length. We denote the length of the orbit by $\mathrm{ord}_f(v)$ and $\mathrm{ord}_f(w)$ and assume w.l.o.g. $2 \leq \mathrm{ord}_f(v) < \mathrm{ord}_f(w)$. Then define $g := f^{\mathrm{ord}_f(v)}$ as an iteration of f. The vertices $v, f(v), ..., f^{\mathrm{ord}_f(v)-1}(v)$ are fixed points of g. Since these are more than one we conclude $g = \mathrm{id}$. But we also see that w is no fixed point of g because otherwise its order is at most $\mathrm{ord}_f(v)$. This is a contradiction. □

2.6 Degree estimations

In this section we will derive some bounds on the maximum degree Δ and average degree $\bar{d}$ of $P(k)$-graphs. The only known bound on the minimal degree is the trivial one: $\delta \geq 2$ from Observation 1.2. We need the following observation on the diameter of $P(k)$-graphs:

Lemma 2.19. *Let G be a $P(k)$-graph for $k \geq 3$. It holds* $\mathrm{diam}(G) \leq \frac{2}{3}k - 3$.

Proof. Let $v, w \in V(G)$ be two vertices. Because G is 2-connected we can use the theorem of Menger (see [6] p. 50 ff.) to conclude that there are two crossing-free paths P_1 and P_2 from v to w. The union $P_1 \cup P_2$ is a cycle C and according to Lemma 2.7 of length at most $\frac{4}{3}k - 6$. Hence, one of the paths P_1 and P_2 is of length at most $\frac{1}{2}|C| = \frac{2}{3}k - 3$. This holds for all pairs of vertices and so we are done. □

Note that this also holds for $k \leq 4$ despite of the fact that $\frac{2}{3}k - 3 < 0$, because we already know that $P(k)$ is the empty set for such values of k. Our next result will be a simple lower bound for the maximum degree Δ of $P(k)$-graphs. Actually the derived bound holds for all graphs of diameter at most $\frac{2}{3}k - 3$. An upper bound is given by $\Delta \leq 2c_{k+1}$ with c_{k+1} the number of $(k+1)$-cycles in G. This follows from the fact that different $(k+1)$-cycles do not share edges and each cycle passes each vertex at most once.

Lemma 2.20. *For the maximum degree Δ of a $P(k)$-graph G holds*

$$\Delta \geq 1 + \left(\frac{n+1}{2}\right)^{\frac{3}{2k-6}} = \Omega\left(n^{\frac{3}{2k-6}}\right). \tag{2.2}$$

Proof. Choose an arbitrary vertex $v \in V$. We estimate the size of the ℓ-neighborhood $N_G^\ell(v)$ with the maximum degree Δ:

$$|N_G^\ell(v)| \leq \Delta(\Delta-1)^{\ell-1}.$$

We know from Lemma 2.19 that G is of diameter at most $\frac{2}{3}k-3$. Therefore every other vertex than v is contained in some ℓ-neighborhood for an $\ell \in \{1, ..., \frac{2}{3}k-3\}$ and there must hold

$$\begin{aligned} n-1 &\leq \sum_{\ell=1}^{2k/3-3} |N_G^\ell(v)| \\ &\leq \Delta \sum_{\ell=1}^{2k/3-3} (\Delta-1)^{\ell-1} \\ &= \Delta \frac{(\Delta-1)^{2k/3-2}-1}{\Delta-2} \\ &= \left(1+\frac{2}{\Delta-2}\right)\left((\Delta-1)^{2k/3-2}-1\right). \end{aligned}$$

We know that $\Delta \geq 4$ and hence deduce $n-1 \leq 2\left((\Delta-1)^{2k/3-2}-1\right)$ which results in statement (2.2). $\square$

For the further observations we need two classical results. The proofs are included because they are short and instructive.

Lemma 2.21. *Let G be a graph of average degree $\bar{d}_G = 2d$ (note: d does not have to be integral). Then there is a subgraph $H \subseteq G$ with minimum degree $\delta_H \geq d+1$ and average degree $\bar{d}_H \geq 2d$.*

Proof. If G has minimum degree $\delta \geq d+1$ we are done. Otherwise we can choose a vertex v_0 of degree $d_0 := d_G(v_0) \leq d$. If we remove v_0 from G we can express the new average degree by

$$\bar{d}_{G-v_0} = \frac{n\bar{d}_G - 2d_0}{n-1}.$$

Using this identity we see that $\bar{d}_{G-v_0} < \bar{d}_G$ implies $\bar{d}_G < 2d_0 \leq 2d$. But this is a contradiction. Hence by removing a vertex of degree at most d the average degree is never decreasing and finally by repeating this procedure we arrive at a graph of minimum degree $\delta_H \geq d+1$ since all vertices of smaller degree are removed. $\square$

Lemma 2.22. *Let G be a graph not containing a k-path. Then the number of edges is bounded by*

$$|E| \leq \frac{k-1}{2}n.$$

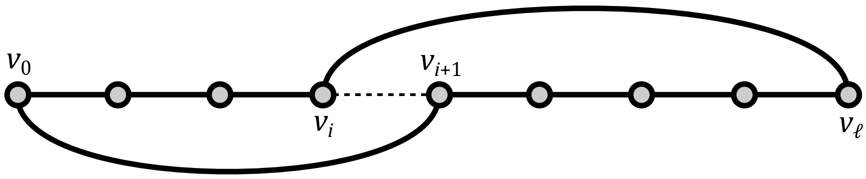

Figure 2.9: A path with the required configuration as used in the proof of Lemma 2.22. The emerging $(\ell+1)$-cycle is emphasized.

Proof. Assume $|E| > \frac{k-1}{2}n$ which implies $\bar{d}_G = \frac{2|E|}{n} > k-1$. According to Lemma 2.21 we can find a subgraph $H \subseteq G$ with minimum degree $\delta_H > \frac{1}{2}(k+1)$ and average degree $\bar{d}_H > k-1$. In particular, if H is not connected we can consider its connected component of the largest average degree. In the case of choosing H to be this component this also guarantees the foregoing degree estimations. Note that H must have at least $k+1$ vertices.

Assume $P = v_0 v_1 \cdots v_\ell$ to be the longest path in H of length $\ell < k$, which then leads to a contradiction as follows: We show that this path can be used to find an $(\ell+1)$-cycle in H. Because H is connected and has at least one more vertex (the path contains at most k vertices) there must be a vertex adjacent to this cycle. If we start in this vertex, go over to the cycle and traverse it in an arbitrary direction, we finally get a path of length $\ell+1$ which is a contradiction to the choice of P. It would follow $\ell \geq k$ and so it remains to construct this $(\ell+1)$-cycle.

Since P cannot be extended, all the neighbors of v_0 and v_ℓ must already be contained in P. So there are more than $\frac{1}{2}(k+1)$ neighbors of v_0 in P. Each one of these has a previous vertex (with respect to the order induced by P). So there are fewer than $\ell - \frac{1}{2}(k+1) < \frac{1}{2}k$ vertices left which are not followed by a neighbor of v_0. These are not enough to cover all neighbors of v_ℓ in P which are more than $\frac{1}{2}(k+1)$ many. Hence there is a pair $v_i, v_{i+1} \in P$ which satisfies that v_i is adjacent to v_ℓ and v_{i+1} is adjacent to v_0. Now $C := v_0 v_1 \cdots v_i v_\ell v_{\ell-1} \cdots v_{i+1} v_0$ is the desired $(\ell+1)$-cycle (see Fig. 2.9) and we are done. □

The last lemma, together with the upper bound $\frac{1}{2}k(k+1) - \frac{15}{8}$ on the path length in $P(k)$-graphs from Corollary 2.11, already shows that for the edge count of $P(k)$-graphs holds $|E| = \mathcal{O}(n)$, thus they indeed do not fit the purpose of representing edge-rich C_{2k}-free graphs as discussed in the introduction. In detail, we can derive

$$|E| \leq \frac{1}{4}\left(k(k+1) - \frac{23}{4}\right) n < \frac{1}{4}k^2 n.$$

So the edge count is linearly bounded above with respect to k^2 and n. The following result will tighten this estimation in the way, that the upper bound is even linear in k.

Lemma 2.23. *Let G be a $P(k)$-graph and C an even cycle of length 2ℓ in G. It holds $\bar{d} < 2(k+\ell-2)$. In detail there holds:*

(i) $\bar{d} < 3k-3$ *for odd k.*

(ii) $\bar{d} < \frac{10}{3}k - 6$ *for even k.*

Proof. The idea is to remove the even cycle C from G. This directly deletes at most $2\ell\Delta$ edges. The remaining graph, according to Lemma 2.8, contains no path of length $2(k-\ell)-3$ or longer. Here we can use the previous Lemma 2.22 and estimate

$$\begin{aligned} |E| &\leq |E(G \setminus C)| + 2\ell\Delta \\ &\leq \frac{[2(k-\ell)-3]-1}{2} n + 2\ell\Delta \\ &= (k-\ell-2)n + 2\ell\Delta. \end{aligned}$$

Because of the identity $\bar{d} = \frac{2|E|}{n}$ and $\Delta < n$ we found the first bound:

$$\bar{d} \leq 2(k-\ell-2) + 4\ell\frac{\Delta}{n} < 2(k-\ell-2) + 4\ell = 2(k+\ell-2).$$

This upper bound is tighter if ℓ is smaller and therefore we try to use the smallest even cycle of G we know of to bound $\bar{d}$. For odd k we know about the even $(k+1)$-cycles, hence we can choose $\ell = \frac{k+1}{2}$ and get (i). For even k we only know the existence of even cycles with length at most $\frac{4}{3}k - 2$ by Lemma 2.7 and so we can only assume $\ell \leq \frac{2}{3}k - 1$ which leads to (ii). □

In Chapter 4 we shall see that some generalizations of $P(k)$-graphs necessarily lead us to regular graphs as only possible solutions. Combining Lemma 2.20 and 2.23 shows that there are at most finitely many regular $P(k)$-graphs for $k \geq 3$. In the case of regular graphs it must hold $\bar{d} = \Delta$. Using the established lower and upper bounds yields

$$1 + \left(\frac{n+1}{2}\right)^{\frac{3}{2k-6}} \leq 3k-3 \qquad \Rightarrow \qquad n \leq 2(3k-4)^{\frac{2k-6}{3}} - 1$$

for odd k and a qualitatively equivalent result for even k. Now we are going to use the connection between edge count and the number of $(k+1)$-cycles to derive

Corollary 2.24. *Let G be a $P(k)$-graph and c_{k+1} the number of $(k+1)$-cycles in G. It holds:*

(i) $c_{k+1} \leq \frac{3}{2}n$ *for odd k.*

(ii) $c_{k+1} \leq \frac{5}{3}n$ *for even k.*

Proof. Since G possesses a decomposition into all its $(k+1)$-cycles that contains every edge in E exactly once, we derive

$$c_{k+1} = \frac{|E|}{k+1} = \frac{\bar{d}n}{2(k+1)}.$$

Using the estimations on $\bar{d}$ from Lemma 2.23 directly leads to above estimations on c_{k+1}. □

In [3] Bondy proved an even better upper bound for odd k only dependent on k: $c_{k+1} \leq \frac{1}{2}(k-1)$. When c_{k+1} is bounded, also the edge count is. But with a limited number of edges, graphs with to many vertices are no longer connected. Hence in the case of odd k there is also an upper bound for n. This implies that there are at most finitely many $P(k)$-graphs for odd k. A lower bound on the number of $(k+1)$-cycles was proven in [10]: $c_{k+1} \geq 4$.

2.7 Summary

In this chapter we collected properties of $P(k)$-graphs for general $k \geq 3$. Some of these results are new, some are known but may were presented with a new proof. This section lists what should mainly be kept in mind from the whole chapter. Of course, these are not the final answers and further results are welcome.

Theorem 2.25. *Let G be a $P(k)$-graph for some $k \geq 3$. The following statements hold:*

(i) G is uniquely decomposable into edge disjoint $(k+1)$-cycles and these are the only $(k+1)$-cycles in G (see Observation 1.2).

(ii) G contains an even cycle of length 2ℓ with $\ell \in \{3, ..., k-5\}$ but no even cycle for $\ell \in \{2, k-4, k-3, k-2, k-1, k\}$ (cf. [11], [16]).

(iii) G contains a cycle of length at least $k+5$, but all cycles are of length at most $\frac{4}{3}k - 2$ (cf. [10]).

(iv) Any two $(k+1)$-cycles in G intersect at least twice. This will be extended to at least five and at most $k-1$ crossings in Chapter 3 (see Lemma 2.12, [3]).

(v) All paths in G are of length at most $\frac{1}{2}k(k+1) - \frac{15}{8}$ (see Corollary 2.11).

(vi) G is 2-connected. Any cut is of even size, hence the edge-connectivity is even (see Section 2.3).

(vii) A proper subgraph of a $P(k)$-graph cannot be a $P(k)$-graph on its own (see Section 2.4).

(viii) *G is either asymmetric or has a rotational symmetry, i.e. an $f \in \mathrm{Aut}(G)$ has at most one fixed point and the orbits of all other vertices are equally long (see Section 2.5).*

(ix) *$\delta \geq 2$, $\Delta = \Omega\left(n^{\frac{3}{2k-6}}\right)$, and $\bar{d} = \mathcal{O}(n)$ (see Section 2.6).*

(x) *For fixed odd k there are only finitely many $P(k)$-graphs (see last paragraph in Section 2.6).*

(xi) *If c_{k+1} is the number of $(k+1)$-cycles in G, it holds $4 \leq c_{k+1} \leq \frac{1}{2}(k-1)$ for odd and $4 \leq c_{k+1} \leq \frac{5}{3}n$ for even k (see Corollary 2.24).*

(xii) *$k \geq 21$ (see [10]).* $\square$

3 The cycle-intersection-conjecture

When one starts studying $P(k)$-graphs a common way to get a feeling for the possibilities and impossibilities of such graphs is trying to construct one. A good point to start a construction is a subgraph we surely know to exist, e.g. a $(k+1)$-cycle. But already installing a second $(k+1)$-cycle presents problems. A bit of experimenting reveals that it is a hard task to configure the cycles that way that they do not contain two k-paths with the same end vertices. Actually, it seems to be impossible if one tries to intersect the cycles at least twice. This observation leads us to the following conjecture which could be used to attack the conjecture of Kotzig.

Conjecture 3.1 (cycle-intersection-conjecture). *Whenever two cycles of length $k+1$ intersect in at least two vertices, their union contains a double k-path.*

This idea was already mentioned in [3] as a possible attack, labeled as worth to examine. If this conjecture turns out to be true this will give a direct proof of the conjecture of Kotzig: all $(k+1)$-cycles have to intersect at least twice (see Lemma 2.12) and $c_{k+1} \geq 4$. We were not able to give a full proof of the conjecture, but we will present some techniques and handle some cases for small and large numbers of crossings or nearly uniform vertex distributions.

Troughout this chapter we will always work in the context of the cycle-intersection-conjecture: the graph G is always a *cycle intersection*, i.e. the union of two edge disjoint $(k+1)$-cycles C_1 and C_2 for $k \geq 3$ which intersect in $s \geq 2$ vertices (the case of not edge disjoint cycles is trivial and was discussed in Observation 1.2). This can look like as illustrated in Fig. 3.1. To avoid confusion we emphasize that some subpaths of e.g. C_1 which are referred to as arcs of C_1 can also be interpreted as chords of C_2 and vice versa as long as their end vertices are crossings of these cycles. So it might be the case that a path P is a chord *of* C_1 but a chord *from* C_2, which means $P \subseteq C_2$. Therefore, most of the time we will just omit to state explicitly the cycle a chord or arc belongs to. This means we write, e.g. "P is a chord" instead of "P is a chord of C_1". This includes that if not mentioned otherwise the word "chord" (or "arc") always refers to a chord (or an arc) of C_1 or C_2. Also, when not referring to a certain cycle, the words "chord" and "arc" can be used interchangeably. We will prefer to call it a chord. So each of the cycles C_1 and C_2 consists of s chords. All this conventions will come in useful especially when introducing the following generalizing terms:

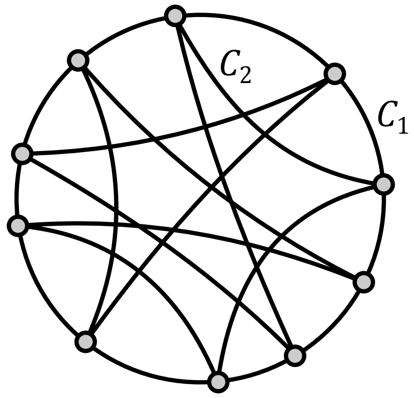

Figure 3.1: An exemplary configuration of the cycles-intersection-conjecture. One of the cycles (here C_1) can be viewed as an actual circle, the other one (here C_2) is represented by a chain of chords of the first one. Note that only the intersection vertices are drawn and there may be inner vertices not included in this simplified illustration.

Definition 3.2. (i) An *inner vertex* is a vertex of G that is not an intersection vertex of the cycles.

(ii) A subpath of C_1 (or C_2) whose end vertices are intersection vertices is called a *near-chord*.

(iii) Let P and P' be a chord and a near-chord from different cycles. If P and P' only intersect in their end vertices, then the cycle $P \cup P'$ is called a *near-loop*.

(iv) A near-loop is called *chord-dominated* if its chord is at least as long as its near-chord.

(v) A near-loop consisting of two chords is called a *loop*.

Note that a chord (of C_1 or C_2) is a special case of a near-chord where only the end vertices are crossings of the cycles. Also the decomposition of a near-loop into a chord and a near-chord is unique. Hence we can speak of the near-loop's chord or near-chord. Figure 3.2 visualizes above definitions. The next definition will be a useful tool in many proofs of this chapter:

Definition 3.3. The *chord graph* $G^c = (V^c, E^c, I)$ of a cycle intersection G is a multi-graph defined by its vertex set V^c and edge set E^c, as well as an incidence relation $I \subseteq V^c \times E^c$:

(i) V^c is the set of intersection vertices of G.

(ii) E^c is the set of chords of G.

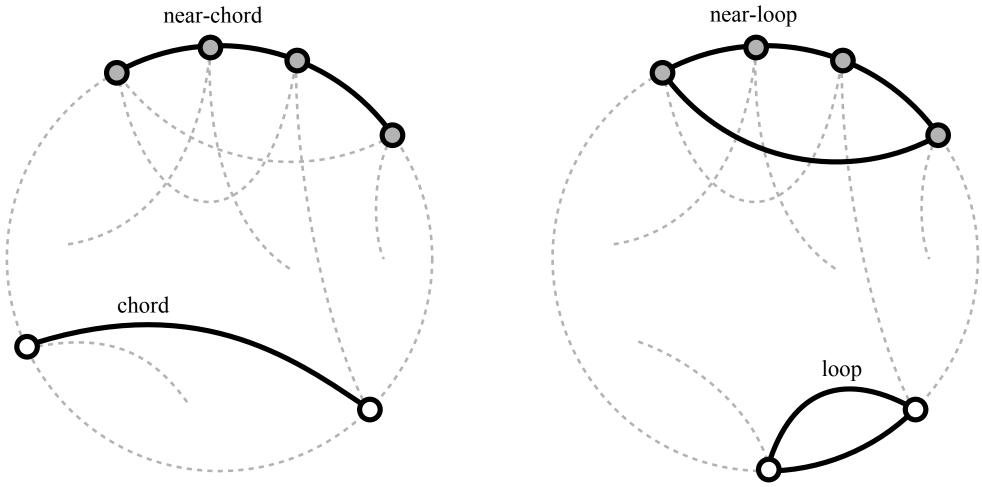

Figure 3.2: On the left side a chord (through the white vertices) and a near-chord (through the gray vertices). On the right side a loop (through the white vertices) and a near-loop (through the gray vertices).

(iii) A vertex $v \in V^c$ and an edge $e \in E^c$ are incident (i.e. $(v, e) \in I$) if v is an end vertex of e.

For $e \in E^c$ the term $|e|$ means the length of the chord e as seen in G.

Observation 3.4. The multi-graph G^c can be seen as emerging from G by just ignoring the inner vertices (see Fig. 3.3). So each former chord now directly connects two vertices. We emphasize that paths and cycles in G^c can also be seen as paths and cycles in G: If S is a path (resp. a cycle) in G^c, then $\bigcup_{e \in E(S)} e$ is a path (resp. a cycle) in G of length $\sum_{e \in E(S)} |e|$. This path (or cycle) in G will also be denoted by S. We also mention that loops of G are multi-edges (i.e. 2-cycles) of G^c. Note that chord graphs have exactly s vertices and are always 4-regular because of the four chords starting in each intersection vertex of G. Also G^c is a *generalized cycle intersection* which is a regular cycle intersection that may contain multiple edges (see the example in Fig. 3.3). G^c consists of two s-cycles intersecting s times.

In the next two sections we will prove the cycle-intersection-conjecture for some values of s. Section 3.3 then generalizes the results of Section 3.1. Although the main result of Section 3.1 is a direct consequence of Lemma 3.18 from the later section, we introduce it separately as it is done in most of the literature (e.g. see [3]). The content of Section 3.3 then is a new result of this thesis.

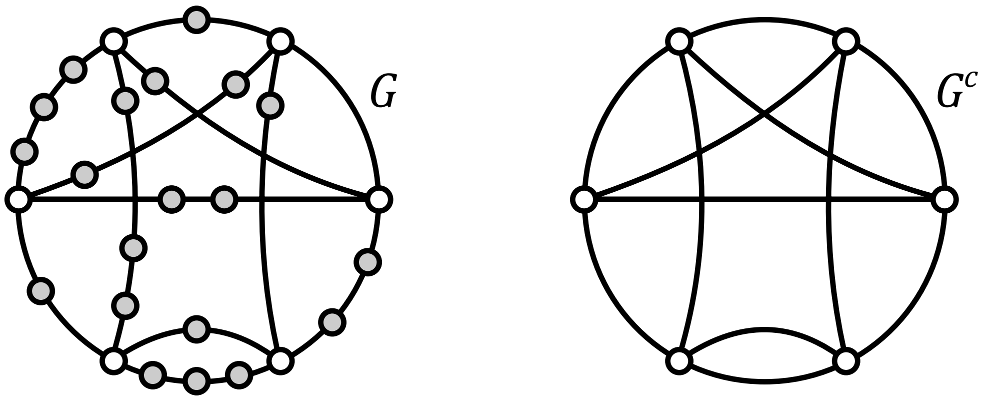

Figure 3.3: The multi-graph on the right side is the chord graph G^c of the graph on the left side. It emerges from G by "forgetting" about the inner vertices. Only the intersection vertices (the white ones) are left. G^c is a generalized cycle intersection because it contains a multiple edge.

3.1 Equally long chords and the cases $s \in \{k, k+1\}$

In the case $s = k + 1$, when all vertices are intersection vertices and there are no inner vertices, the graph $G := C_1 \cup C_2$ has exactly $k + 1$ vertices and is isomorphic to its chord graph G^c. Here the cycles C_1 and C_2 are two Hamiltonian cycles of G and a k-path is necessarily a Hamiltonian path. Consequently we are interested in finding two Hamiltonian paths with the same end vertices. A Hamiltonian path can be obtained by deleting an edge from a Hamiltonian cycle. If we know about two Hamiltonian cycles sharing an edge, we can remove this common edge and each of the cycles becomes a k-path, both with the same end vertices. Unfortunately the already known Hamiltonian cycles C_1 and C_2 are edge disjoint. Thus, the hard part in performing this idea is finding a Hamiltonian cycle C_H containing edges from both cycles C_1 and C_2. For this purpose we need the term of Hamiltonian decomposition and a powerful theorem on 4-regular graphs which we are not going to prove here.

Definition 3.5. Let G be a graph. A *Hamiltonian decomposition* of G is a set of edge disjoint Hamiltonian cycles of G whose union is the whole graph again.

Theorem 3.6 (Thomason, [15])**.** *Let G be a 4-regular (multi-)graph and $e_1, e_2 \in E(G)$ two edges. There is an even number of Hamiltonian decompositions of G. Further there is an even number of Hamiltonian decompositions of G that assign e_1 and e_2 to the same cycle.*

Observation 3.7. The statement of Theorem 3.6 will be used in the following different way: If G is 4-regular and $e_1, e_2 \in E(G)$ are two edges, then there is an even number of Hamiltonian decompositions of G that assign e_1 and e_2 to different cycles.

Since the total number of Hamiltonian decompositions is even and the part that assign e_1 and e_2 to the same cycle is of even size too (see Theorem 3.6), their difference is also even. This was the number we are looking for.

Corollary 3.8. *For a (generalized) cycle intersection G with $s = k + 1$ there is a Hamiltonian cycle in G that contains edges from C_1 and from C_2.*

Proof. Note that G is 4-regular because it is isomorphic to its 4-regular chord graph G^c (see Observation 3.4). Choose $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$. The pair (C_1, C_2) is a Hamiltonian decomposition of G with e_1 and e_2 in different Hamiltonian cycles. According to Observation 3.7 there must be at least one more decomposition into Hamiltonian cycles C_1' and C_2'. Of course, C_1' (resp. C_2') has to contain edges from C_1 and C_2 because otherwise (C_1', C_2') equals the decomposition (C_1, C_2). □

As discussed above this solves the case $s = k + 1$. This can be extended to the following case:

Lemma 3.9. *If the chords of C_1 and C_2 are all of the same length ℓ with at most one exception per cycle, then G contains a double k-path.*

Proof. Let $e_1 \subseteq C_1$ and $e_2 \subseteq C_2$ be the two chords of lengths different from ℓ, or arbitrary chords if all chords are of the same length. Consider the chord graph G^c of G and note that $e_1, e_2 \in E^c$. As mentioned in Observation 3.4 G^c is a generalized cycle intersection of s-cycles, also called C_1 and C_2, with s crossings. So we can use Corollary 3.8 to construct a Hamiltonian cycle C_H in G^c containing e_1 but not e_2 and also different from C_1. In G^c the cycle C_H is of length s. Because at most e_1 has a length different from ℓ we observe a length of $(s-1)\ell + |e_1| = |C_1| = k + 1$ for C_H in G. Also C_1 and C_H share the chord e_1. So removing a single edge e of e_1 in G yields two paths of length k between the end vertices of e and we are done. □

Lemma 3.9 includes the case $s = k$: all chords are single edges except for two which are of length two.

3.2 Loops and the cases $s \in \{2, 3, 4\}$

For the cases $s \in \{2, 3, 4\}$ we need to have a closer look at the influence of loops on the path structure of G. We will see that both, even and odd loops, may imply a double k-path. The fact that for small s all possible cycle intersections have such loop configurations proves the cycle-intersection-conjecture for these special cases. The case $s = 2$ is treated separately.

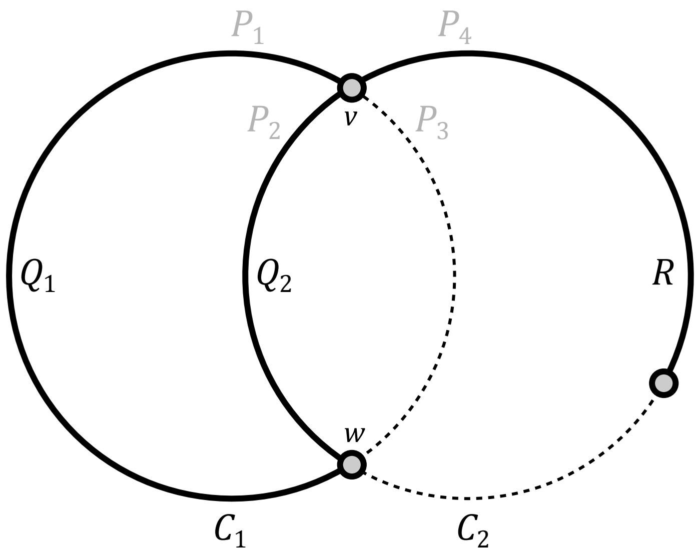

Figure 3.4: Configuration used in the proof of Lemma 3.10, two $(k+1)$-cycles intersecting twice.

Lemma 3.10. *If $s = 2$ there is a double k-path.*

Proof. Let $v, w \in V(C_1) \cap V(C_2)$ be the intersection vertices. Observe that the arcs of C_1 and C_2 between these vertices are four pairwise crossing-free paths $P_i, i = 1, 2, 3, 4$ between v and w. W.l.o.g. $|P_1| \geq |P_2| \geq |P_3| \geq |P_4|$. Because the lengths of two arcs from the same cycle sum up to $k+1$ it holds

$$|P_1| + |P_4| = k+1, \qquad |P_2| + |P_3| = k+1, \qquad |P_1|, |P_2| \geq \frac{k+1}{2}.$$

We know that the union of two of the three paths P_1, P_2 and P_3 gives an even cycle (see Lemma 2.2). Also all three paths are of length at most k, hence the cycle is of length at most $2k$.[1] Call the two paths forming this cycle Q_1 and Q_2 and the remaining path R (see Fig. 3.4). Note that R is a chord of the cycle $Q_1 \cup Q_2$. Because of the considerations on even cycles in Observation 2.1 it must hold

$$\begin{aligned} k &\geq \frac{1}{2}(|Q_1| + |Q_2|) + |R| \\ &= \frac{1}{2}(|Q_1| + |Q_2| + |R|) + \frac{1}{2}|R| \\ &= \frac{1}{2}(|P_1| + |P_2| + |P_3|) + \frac{1}{2}|R| \\ &= \frac{1}{2}|P_1| + \frac{k+1}{2} + \frac{1}{2}|R|. \end{aligned}$$

[1]Note that we have to check this because we no longer can use the statements of Lemma 2.7 which is only valid in the context of $P(k)$-graphs.

But this rearranges to $|R| + 2 \leq k + 1 - |P_1| = |P_4|$ which is impossible because P_4 is the shortest of these paths. $\square$

Lemma 3.11. *If G contains an even loop then also a double k-path.*

Proof. Let C be the loop of length 2ℓ and observe that $2 \leq \ell < k$. Let $P_1 \subseteq C_1$ and $P_2 \subseteq C_2$ be the two chords the loop consists of. Assume w.l.o.g. $|P_1| \geq |P_2|$, hence $|P_2| \leq \ell$. P_2 is an arc of C_2, so let $P_2' \subseteq C_2$ be the other arc which now is a chord of C. It holds $|P_2'| \geq k + 1 - \ell$. But according to Lemma 2.5 a chord of C of length at least $k - \ell$ implies a double k-path. $\square$

Observation 3.12. Note that the last result for even loops also holds for chord-dominated even near-loops. Using the terms of the proof of Lemma 3.11, P_1 would be a chord. So P_2' is C-crossing-free and the proof works as before.

Observation 3.13. From Observation 2.1 we know that an even cycle potentially implies a double k-path. But there is a similar mechanism working for odd cycles if there are two of them. Of course, an odd cycle cannot be divided into arcs of the same length, but the idea is to compensate this difference by a corresponding choice of the arcs in the second odd cycle. The approach is essentially illustrated in Fig. 3.5.

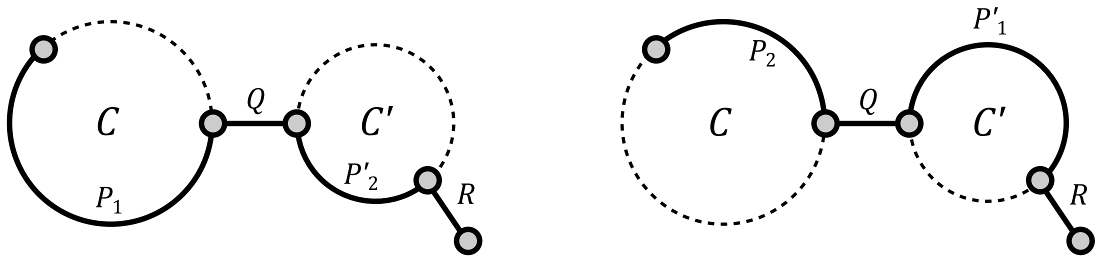

Figure 3.5: A pair of cycles of matching parity may induce a double k-path. The chords P_1, P_2, P_1' and P_2' are chosen that way, that it holds $|P_1| + |P_2'| = |P_1'| + |P_2|$.

The two odd cycles C and C' are divided into the arcs $P_1, P_2 \subseteq C$ and $P_1', P_2' \subseteq C'$. The cycles are connected by a path Q and there is an additional path R starting in one of the cycles, say in C', and not intersecting the cycles again. The important part is that the arcs must be chosen that way, so that $|P_1| + |P_2'| = |P_2| + |P_1'|$. Since the arcs of C' are predefined by its crossings with Q and R, there are some restrictions on the length of C. In general there are the following difficulties in finding such configurations:

1. The cycle C has to be long enough to compensate the encountered difference in the arc lengths of C'. This is always satisfied if $|C| \geq |C'|$.

2. The path Q must not be too long because otherwise a k-path might not completely pass around the second cycle. In detail it must hold $|Q|+|P_1|+|P_2'| \le k$ or equivalent $|Q| + |P_2| + |P_1'| \le k$.
3. The path R must not be too short because otherwise there might be not enough space to accomplish a single k-path. In detail it must hold $|R|+|Q|+|P_1|+|P_2'| \ge k$ or equivalent $|R| + |Q| + |P_2| + |P_1'| \ge k$.

Note that this technique not only works for two odd cycles, but also if the both cycles are even. The only important fact are the matching parities of the cycle lengths.

Although these are a lot of things to care about we will see that such a configuration can be found if there are two odd loops in G. The proof of the following lemma is a bit long because we have to check the three conditions from above but it mainly follows Observation 3.13.

Lemma 3.14. *If G contains two odd loops then also a double k-path.*

Proof. We try to find a configuration as discussed in Observation 3.13 (see Fig. 3.6). The two odd cycles are represented by the odd loops C and C'. Choose $|C| \ge |C'|$. Call the chords that the loops consist of $P_1, P_2 \subseteq C$ and $P_1', P_2' \subseteq C'$ where P_1 and P_1' are from C_1 and P_2 and P_2' are from C_2. W.l.o.g. assume

$$|P_1| + |P_1'| \ge |P_2| + |P_2'|. \tag{3.1}$$

The paths P_2 and P_2' are arcs of C_2 and between these there are two more arcs Q and R of C_2, possibly of length zero. Choose $|R| \ge |Q|$. If $|Q| = |R| = 0$ we are in the case $s = 2$ in which we know of the existence of a double k-path (see Lemma 3.10). Hence assume $|R| \ge 1$. Define the two edge disjoint arcs $\tilde{P}_1$ and $\tilde{P}_2$ of C, both starting in the same vertex v and of length

$$|\tilde{P}_1| = \frac{1}{2}(|P_1| + |P_2| + |P_1'| - |P_2'|), \qquad |\tilde{P}_2| = \frac{1}{2}(|P_1| + |P_2| - |P_1'| + |P_2'|).$$

Observe that these lengths are positive integers according to (3.1) and the odd length of the loops. These paths indeed satisfy the following important conditions:

(i) The lengths of $\tilde{P}_1$ and $\tilde{P}_2$ sum up to $|C|$, hence it holds $|\tilde{P}_1| + |\tilde{P}_2| = |P_1| + |P_2|$. These arcs decompose C and are both between the same end vertices.

(ii) The identity $|\tilde{P}_1| + |P_2'| = |\tilde{P}_2| + |P_1'|$ is satisfied, thus the different lengths of the arcs of both loops compensate perfectly.

We are done building the configuration of Observation 3.13. It remains to check the two requirements on the paths Q and R:

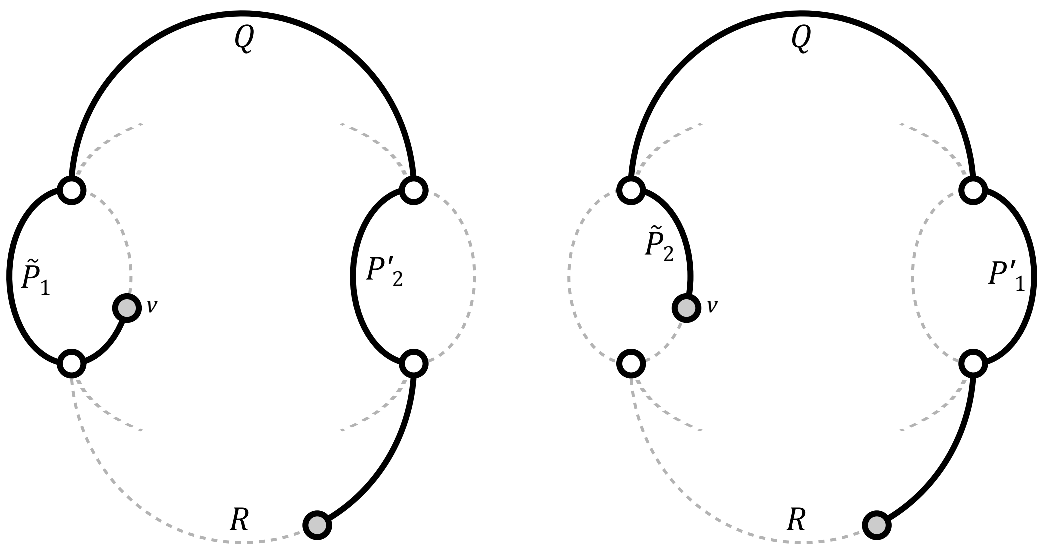

Figure 3.6: Configuration used in the proof of Lemma 3.14. This applies the idea discussed in Observation 3.13 and constructs the two k-paths between the gray vertices.

Step 1: Assume Q is too long, i.e. $|Q| + |\tilde{P}_1| + |P_2'| > k$ or equivalent $|Q| + |\tilde{P}_2| + |P_1'| > k$. Consider the sum of these inequalities and assume at first that $|Q| = |R|$. We obtain

$$\begin{aligned} 2k &< 2|Q| + |\tilde{P}_1| + |\tilde{P}_2| + |P_2'| + |P_1'| \\ &= 2|Q| + |P_1| + |P_2| + |P_2'| + |P_1'| \\ &= |Q| + |R| + |P_2| + |P_2'| + |P_1| + |P_1'| \\ &= k + 1 + |P_1| + |P_1'| \end{aligned}$$

which can be rearranged to $|P_1| + |P_1'| > k - 1$ or $|P_1| + |P_1'| \geq k$. This implies that the loops C and C' must intersect, therefore $|Q| = 0$. But because of our assumption $|R| = |Q|$ this contradicts $|R| \geq 1$. So it must hold $|R| > |Q|$. We proceed in the same way:

$$\begin{aligned} 2k &< 2|Q| + |\tilde{P}_1| + |\tilde{P}_2| + |P_2'| + |P_1'| \\ &= 2|Q| + |P_1| + |P_2| + |P_2'| + |P_1'| \\ &\leq |Q| + |R| + |P_2| + |P_2'| + |P_1| + |P_1'| - 1 \\ &= k + |P_1| + |P_1'|. \end{aligned}$$

This implies $|P_1| + |P_1'| > k$ or $|P_1| + |P_1'| = k + 1$. But then again $|R| = 0$ and so we know $|Q| + |\tilde{P}_1| + |P_2'| \leq k$.

Step 2: Assume R is too short, i.e. $|R| + |Q| + |\tilde{P}_1| + |P_2'| < k + 1$ or equivalently

$|R| + |Q| + |\tilde{P}_2| + |P'_1| < k + 1$. Again consider the sum of the inequalities:

$$\begin{aligned} 2k + 2 &> 2|R| + 2|Q| + |\tilde{P}_1| + |\tilde{P}_2| + |P'_2| + |P'_1| \\ &= 2|R| + 2|Q| + |P_1| + |P_2| + |P'_2| + |P'_1| \\ &= 2(|R| + |Q| + |P_2| + |P'_2|) + |P_1| + |P'_1| - (|P_2| + |P'_2|) \\ &= 2k + 2 + |P_1| + |P'_1| - (|P_2| + |P'_2|). \end{aligned}$$

This is equivalent to $|P_1| + |P'_1| < |P_2| + |P'_2|$. This contradicts our assumptions. So also the condition $|R| + |Q| + |\tilde{P}_1| + |P'_2| \geq k + 1$ is satisfied.

Continuing along the ideas of Observation 3.13, we now can construct a double k-path. Take the path $R' \subseteq R$ of length $k - |Q| - |\tilde{P}_1| - |P'_2| = k - |Q| - |\tilde{P}_2| - |P'_1|$. The two paths

$$S_1 := \tilde{P}_1 Q P'_2 R', \qquad S_2 := \tilde{P}_2 Q P'_1 R'$$

are of length k and connect the same vertices. To demonstrate the procedure we calculate the length of S_1:

$$\begin{aligned} |S_1| &= |\tilde{P}_1| + |Q| + |P'_2| + |R'| \\ &= |\tilde{P}_1| + |Q| + |P'_2| + k - |Q| - |\tilde{P}_1| - |P'_2| \\ &= k. \end{aligned}$$

□

Observation 3.15. Also for this result there holds a similar generalization as in Observation 3.12. Again, we are going to use the terms of the proof of Lemma 3.14. Assume C and C' to be only near-loops. If the sum of the lengths of the chords exceeds or equals the sum of the lengths of the near-chords, which implies P_1 and P'_1 to be the chords, the paths Q and R are C- and C'-crossing-free. The rest of the proof proceeds as above.

This yields the main result of this section.

Corollary 3.16. *For $s \in \{2, 3, 4\}$ there is always a double k-path.*

Proof. The case $s = 2$ is treated by Lemma 3.10. Observe that the only possible configurations with three or four crossings are shown in Fig. 3.7. All these configurations contain two loops. Either one of the loops is even or both loops are odd. In both cases we know of the existence of a double k-path by Lemma 3.11 and 3.14. □

Figure 3.7 and 3.8 also contain the left intersection configurations not treated completely by above results for $s \in \{5, 6\}$. Indeed, Kotzig proved in [12] that these configurations also induce double k-paths. This yields $7 \leq s \leq k - 1$ and hence another proof of the conjecture of Kotzig for $k \leq 7$.

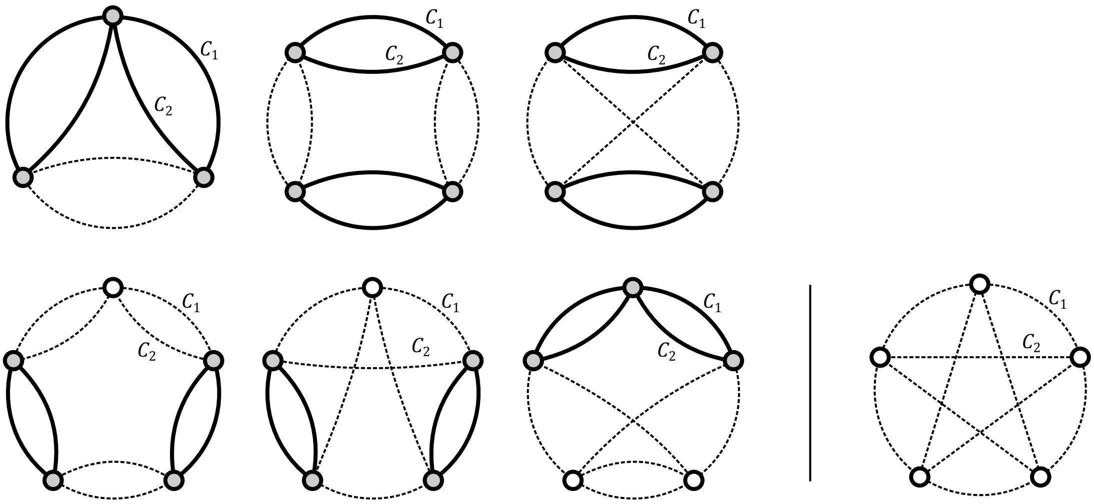

Figure 3.7: All possible intersection configurations for the cases $s \in \{3, 4, 5\}$. The only configuration not containing two loops is the K_5-configuration shown on the right.

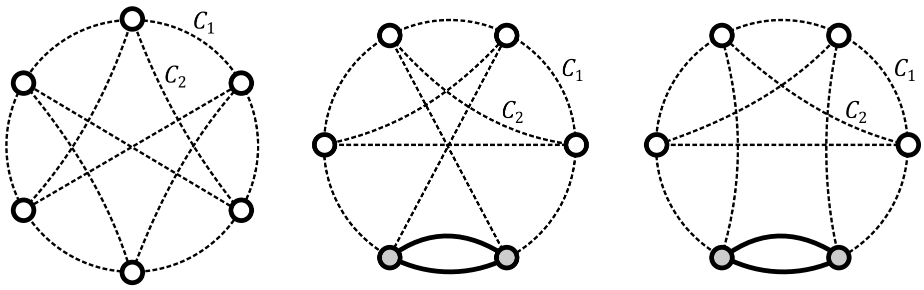

Figure 3.8: These are the only three out of ten possible configurations for $s = 6$ that do not directly imply a double k-path by Lemma 3.11 and 3.14. The loops in the middle and right configuration have to be odd. Also these two configurations are isomorphic: the first one emerges from the second one by exchanging C_1 and C_2 and vice versa.

3.3 Nearly equally long chords

In this section we are going to extend the idea of Section 3.1 where we discussed $s \in \{k, k+1\}$. In general, when considering the chord graph of a cycle intersection G, we always can find two Hamiltonian cycles C_1^H and C_2^H in G^c sharing an edge as we did in Corollary 3.8. Now, these cycles can be used to construct double k-paths in G as we worked out in Lemma 3.9. Problems arise when the cycles C_1^H and C_2^H are not of length $k+1$ in G. We have to find a way to match and correct the cycle lengths. We start by introducing another interesting theorem by Thomason:

Theorem 3.17 (Thomason, [15])**.** *Let G be a $2m$-regular (multi-)graph on at least three vertices for an $m \geq 1$. If G has a Hamiltonian decomposition then there are at least $3^{m-1}(m-1)!$ Hamiltonian decompositions.*

For the 4-regular (multi-)graphs considered in this chapter this gives us the existence of three Hamiltonian decompositions (hence, even four by Theorem 3.6). We can prove

Lemma 3.18. *Let L be the length of the longest and ℓ be the length of the shortest chord in a cycle intersection G. If there holds $sL - \ell \leq k$, then G contains a double k-path.*

Proof. Assume $sL - \ell \leq k$ and consider the chord graph G^c of G. According to Theorem 3.17 we can find three Hamiltonian decompositions of G^c. These cycles as subgraphs of G are of lengths between $s\ell$ and sL. Note that the two cycles from the same decomposition in G^c also cover G. We now return to G and forget about G^c. Because there are exactly $2(k+1)$ edges in G, the lengths of the two cycles in a decomposition are of the same parity. So we can speak of *the parity* of a decomposition. Since we have three decompositions, we can choose two of the same parity. Let C_1^H and C_2^H be the two longer cycles from these two decompositions. Also assume $|C_1^H| \geq |C_2^H|$. Again, the lengths of the two cycles from the same decomposition sum up to $2(k+1)$. Hence we know $|C_1^H|, |C_2^H| \geq k+1$. Note that these cycles have to share a chord because otherwise they would belong to the same decomposition. More precisely, we are looking for a common chord P incident on at least one end vertex to a not common chord, a situation illustrated in Fig. 3.9. This is always possible, because there are common and not common cords and some of them have to overlap in their end vertices. Let $R \subseteq C_1^H$ be such a chord not contained in C_2^H but sharing an end vertex with P. Further, let v and w be the end vertices of P. Choose w to be the end vertex shared with R. Let P' be a subpath of P starting in v and of length $k + |P| - \frac{1}{2}(|C_1^H| + |C_2^H|)$. This is possible because

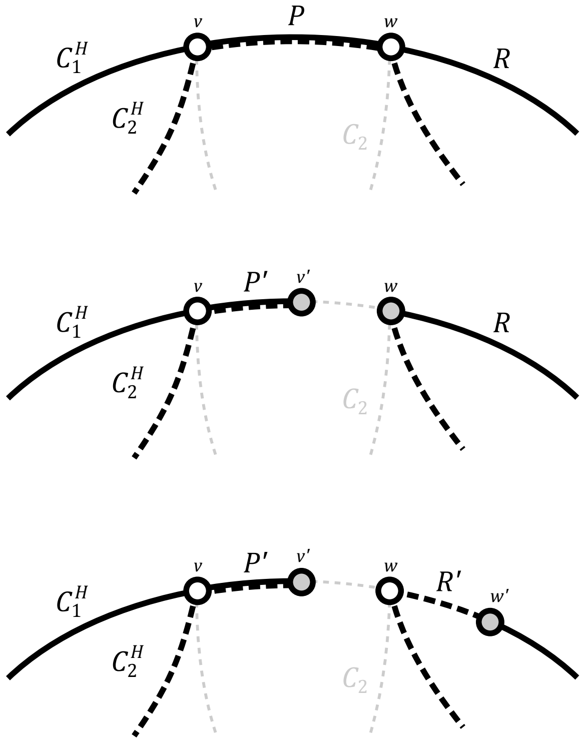

Figure 3.9: The three steps of creating a double k-path in the proof of Lemma 3.18. At first replace the chord P by a subpath P' to achieve the right total length of the paths. The second step equals the path lengths which are now both k. The gray vertices are the common end vertices of the paths.

$|C_1^H|$ and $|C_2^H|$ are of the same parity and the so defined length $|P'|$ satisfies

$$\begin{aligned}|P'| &\geq k + \ell - sL \\ &= k - (sL - \ell) \geq 0\end{aligned}$$

and

$$\begin{aligned}|P'| &\leq k + |P| - (k+1) \\ &= |P| - 1.\end{aligned}$$

Let v' be the other end vertex of P' next to v. Since P is also an arc of both the cycles C_1^H and C_2^H, we can choose the opposed arcs $Q_1 \subset C_1^H$ (with respect to C_1^H) and $Q_2 \subset C_2^H$ (with respect to C_2^H). Observe that $2|P'| + |Q_1| + |Q_2| = 2k$. Define $d := \frac{1}{2}(|C_1^H| - |C_2^H|)$ which is an integer because of the matching parities of the cycles. Consider the subpath R' of R starting in w and of length d. Again, this is possible because

$$\begin{aligned} d &\leq \frac{1}{2}(sL - (k+1)) \\ &\leq \frac{1}{2}(\ell - 1) < \ell. \end{aligned}$$

Let w' be the other end vertex of R' next to w. The two paths $S_1 := w'Q_1vP'$ and $S_2 := w'R'wQ_2vP'$ of length k both connect v' and w'. Note that the part $w'Q_1v$ in S_1 denotes the subpath of Q_1 between w' and v. We show that the lengths are k:

$$\begin{aligned} |S_1| &= |Q_1| - |R'| + |P'| \\ &= |Q_1| - \left[\frac{1}{2}(|C_1^H| - |C_2^H|)\right] + \left[k + |P| - \frac{1}{2}(|C_1^H| + |C_2^H|)\right] \\ &= |Q_1| + k + |P| - |C_1^H| = k. \end{aligned}$$

$$\begin{aligned} |S_2| &= |R'| + |Q_2| + |P'| \\ &= \left[\frac{1}{2}(|C_1^H| - |C_2^H|)\right] + |Q_2| + \left[k + |P| - \frac{1}{2}(|C_1^H| + |C_2^H|)\right] \\ &= |Q_2| + k + |P| - |C_2^H| = k. \end{aligned}$$

□

The statement of Lemma 3.18 gives rise to a more transparent result. Assume $\bar{\ell}$ to be the average chord length. Let ℓ_i be the length of the i-th chord in an arbitrary enumeration of the chords, then

$$\bar{\ell} = \frac{1}{2s}\sum_i \ell_i = \frac{2(k+1)}{2s} = \frac{k+1}{s}.$$

Assume that the length of each chord is between $\bar{\ell} - \epsilon$ and $\bar{\ell} + \epsilon$ with $\epsilon \geq 0$. So the chord lengths differ at most by 2ϵ.

Corollary 3.19. *Using above notations, if there holds*

$$\epsilon \leq \frac{\bar{\ell} - 1}{s+1} = \frac{k+1-s}{s(s+1)},$$

then G contains a double k-path.

Proof. Let L be the length of the longest and ℓ be the length of the shortest chord in the cycle intersection. It holds

$$\begin{aligned} Ls - \ell &\leq (\bar{\ell} + \epsilon)s - (\bar{\ell} - \epsilon) \\ &= \bar{\ell}(s-1) + \epsilon(s+1) \\ &\leq \bar{\ell}(s-1) + \bar{\ell} - 1 \\ &= \bar{\ell}s - 1 = k. \end{aligned}$$

Now, using Lemma 3.18 gives rise to a double k-path. □

This shows that cycle intersections in $P(k)$-graphs cannot have nearly equally long chords. There must always be a chord longer that $\bar{\ell} + \epsilon$ or shorter than $\bar{\ell} - \epsilon$, with $\epsilon = \frac{\bar{\ell}-1}{s+1}$. The restriction is stronger, i.e. ϵ is larger, when there are only a few crossings of the cycles, hence when s is small.

3.4 Ideas for a general approach

In this part we will discuss some techniques which might be useful in finding a general proof for the Conjecture 3.1. Unfortunately, at the moment each one of these ideas is too hard to be caried out in general. Using restrictions on the structure of the cycle intersection then prevents us from finding a proof that fits all the cases.

Induction on the number s of crossings

Induction on the number s of crossings is possible in both directions, i.e. there is a possible induction base for rising and falling s. On the one hand we know that Conjecture 3.1 is true for small $s \leq 6$, on the other hand we also know this for large values $s \geq k$. A possible induction step may be performed by the vertex detachment operation as seen in Fig. 3.10. This decreases the intersection count by one. Also the reversal of the detachment process is conceivable by gluing a vertex of C_2 to C_1. This increases the intersection count s. Problems arising with this approach are the following: when detaching a vertex it must be ensured that the now possible k-paths do not pass both vertices, $\tilde{v}_1$ and $\tilde{v}_2$ (see Fig. 3.10 for the notation). Otherwise, the gluing back would create self-intersections in the paths. Likewise, when first gluing a vertex of C_2 to C_1, a new k-path must not change the cycle at the gluing vertex. Otherwise, detaching the vertex again will destroy the path.

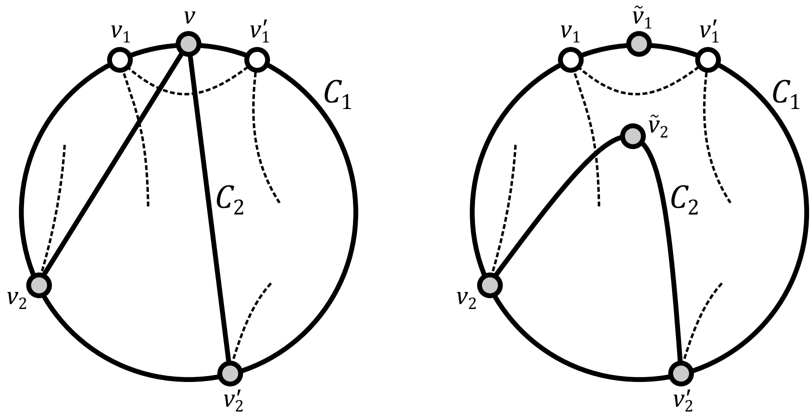

Figure 3.10: The vertex v is detached by replacing it with two vertices $\tilde{v}_1$ and $\tilde{v}_2$ and afterwards joining these to the former neighbors of v: $\tilde{v}_1$ to v_1 and v_1', $\tilde{v}_2$ to v_2 and v_2'. The possibility of emerging multiple edges must be taken into account, but these are easy to deal with.

Induction on the number of inner vertices

The induction base is given by the case $s = k + 1$ and is solved in Section 3.1. An induction step can be performed by removing an inner vertex from both cycles until all vertices are intersection vertices. By following this approach it is hard to control the lengths of the constructed paths. Consider a double k-path after the removal of two inner vertices. When now re-adding these vertices it is possible that the two paths are no longer of the same length. E.g. the first path passed one, the second path passed two chords with now additional vertices. Now, the paths are of length $k + 1$ and $k + 2$. To be precise, both paths have to pass exactly one chord that lost a vertex during the induction step.

Induction on the length of chords and vertex distributions

For this approach a step consists of shifting an inner vertex from one chord to another on the same cycle. This modifies the distribution of the vertices and can either generate a mostly uniform inner vertex distribution or shift all inner vertices to a single chord. The last case is solved as a special case in Section 3.1. Also the completely uniform distribution of the inner vertices is solved, but this solves as induction base only if s divides $k + 1$. In general, the best we can achieve are chords varying in length by only one vertex. Setting $\epsilon = 1$ in Corollary 3.19 tells us that this is solved when $s \leq \bar{\ell} - 2$, or equivalent $s(s + 2) \leq k + 1$. Furthermore, we are facing the same problems as by induction on the number of inner vertices: it is hard to control the length of paths. Shifting vertices back and forth can change the length

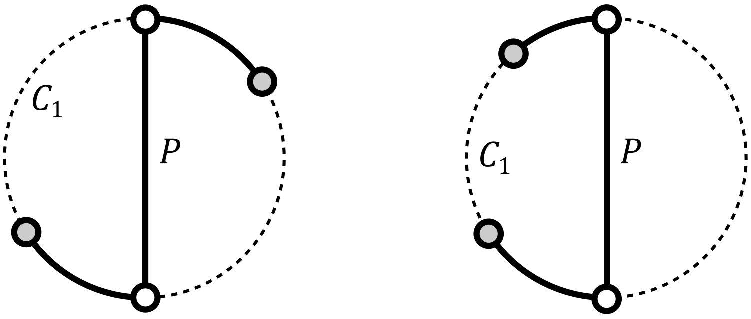

Figure 3.11: The two possible ways a k-path may embedded into $C_1 \cup P$ when it should contain P but starts and ends in C_1.

of paths uncontrollably.

Forbidden near-loops and other substructures

The proofs of Lemma 3.11 and 3.14 can be generalized to near-loops instead of loops. The even near-loops have to be chord-dominated. For the odd ones it is important that the chords are from the same cycle and their total length dominates the total length of the near-chords. This was already discussed in Section 3.2. This gives some restrictions on the form of the cycle intersection. Maybe this can be extended to cover a whole class of forbidden substructures where we can prove that one of these structures is necessarily contained in $C_1 \cup C_2$.

Counting k-paths

Each chord P of C_1 induces a certain number of k-paths where each one contains P but starts and ends in C_1 (see Fig. 3.11). When avoiding double k-paths, all chords together can induce at most $\binom{k+1}{2}$ such paths. Otherwise there would be two with the same end vertices in C_1. Only considering k-paths through single chords gives a number of k-paths linear in k (see [3]). But $\binom{k+1}{2}$ growth like k^2. Hence, it may be necessary to include paths through pairs, triples or general tupels of chords. This technique can used to achieve bounds on s or chord lengths. At the moment it seems very complicated to count such paths without some additional restrictions on the structure of the cycle intersection. In [3] Bondy used this technique to bound the number c_{k+1} of $(k+1)$-cycles in $P(k)$-graphs for odd k.

4 Generalizations

4.1 $P_\ell(k)$-graphs

The most natural generalization of the definition of $P(k)$-graphs may be the following.

Definition 4.1. A graph G on $n \geq 2$ vertices is called a $P_\ell(k)$*-graph* for some $k, \ell \in \mathbb{N}$ if for any two distinct vertices $v, w \in V$ there are exactly ℓ distinct paths of length k between these vertices. The set of all $P_\ell(k)$-graphs will be denoted by $P_\ell(k)$.

Of course, the $P(k)$-graphs are just the $P_1(k)$-graphs. There are no $P_\ell(1)$-graphs for any $\ell \geq 2$ because this would require to include multi-graphs. Again the empty graph and the graph on a single vertex are excluded in the definition above because they would trivially satisfy the path condition. In contrast to the case $\ell = 1$ where graphs decompose into edge disjoint $(k+1)$-cycles, it is not easy to determine a necessary substructure for $P_\ell(k)$-graphs when $\ell \geq 2$. At least we can see, that each edge $e \in E$ is contained exactly ℓ cycles of length $k+1$. Much is known about $P_\ell(2)$-graphs and we will discuss them in Section 4.1.1. For the other cases there are many open questions. In contrast to the special case of $P(k)$-graphs Kotzig found examples for the generalized definition (see [12]).

Observation 4.2. Consider the complete graph K_n for some $n \geq 2$. Choose $k \in \mathbb{N}$ with $k \leq n-1$ and two vertices $v, w \in V(K_n)$. There are exactly

$$\ell := (n-2)^{\underline{k-1}} = \frac{(n-2)!}{(n-k-1)!}$$

k-paths from v to w in K_n. Therefore K_n is a $P_\ell(k)$-graph. The following other examples are known (also see [12]):

1. Take an arbitrary number of K_5's and join them in a common vertex. This yields a $P_6(3)$-graph. It seems that this only works with K_5 (and with K_3 as seen in Section 1.3) but this is still unproven.

2. The octahedron graph is a $P_8(5)$-graph. This graph is isomorphic to the complete 3-partite graph $K_{2,2,2}$. This gives rise to the question which other complete r-partite graphs are $P_\ell(k)$-graphs.

The original motivation for our study of $P(k)$-graphs can directly be applied to $P_\ell(k)$-graphs in the following sense: instead of avoiding C_{2k}-subgraphs we are know considering edge-rich graphs without the corresponding subgraphs illustrated in Fig. 4.1. These graphs are $\ell + 1$ crossing-free k-paths joined in their end vertices. They are bipartite, the partition classes are highlighted in Fig. 4.1. For $\ell = 2$ we observe that these forbidden bipartite subgraphs are just the complete bipartite graphs $K_{2,\ell+1}$. The $P_\ell(k)$-graphs emerge from the requirement to include the maximal possible number of ℓ-paths between any two vertices of G.

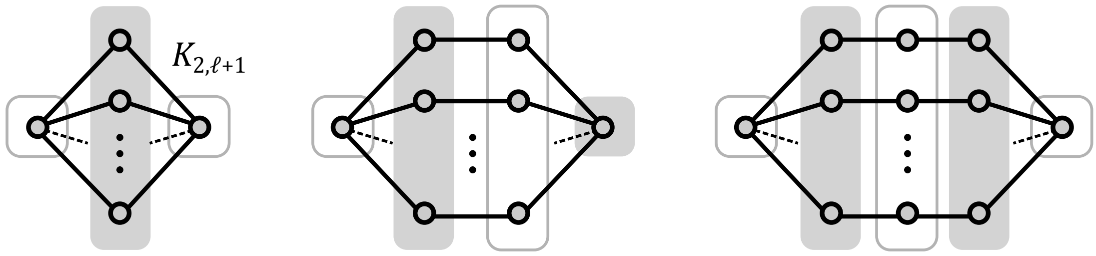

Figure 4.1: Forbidden subgraphs in $P_\ell(2)$-, $P_\ell(3)$- and $P_\ell(4)$-graphs (from left to right). For $k = 2$ these are just the complete bipartite graphs $K_{2,\ell+1}$.

There is not much known about for which pairs of parameters $k, \ell \in \mathbb{N}$ there exists a $P_\ell(k)$-graph.

4.1.1 $P_\ell(2)$-graphs

As discussed above a $P_\ell(2)$-graph is $K_{2,\ell+1}$-free. A main result of this section will be to show that these graphs are indeed edge extremal with this property. The downside is that we can show that there are only finitely many $P_\ell(2)$-graphs for any $\ell \geq 2$. Actually, we also do not know an explicit construction algorithm for arbitrary ℓ, hence do not know about the existence of such graphs besides the ones thar are already listed. A discussion on these graphs can be found in [5].

A big advantage of the case $\ell \geq 2$, in contrast to $\ell = 1$, is that we can directly prove that we only have to consider regular graphs. Again, note that the path condition for $P_\ell(2)$-graphs can alternatively be formulated as follows: Any two distinct vertices $v, w \in V$ have exactly ℓ common neighbors.

Theorem 4.3. *Let G be a $P_\ell(2)$-graph. If $\ell \geq 2$ then G is regular.*

Proof. Since $n \geq 2$ and G is connected there is an edge $e = \{a, b\}$. Define the sets

$$C := N_G(a) \cap N_G(b), \qquad A := N_G(a) \setminus (C \cup \{b\}), \qquad B := N_G(b) \setminus (C \cup \{a\}),$$

so C contains the common neighbors of a and b, A the neighbors of a that are not neighbors of b (except of b itself) and B the neighbors of b that are not neighbors of a (except of a itself, see Fig. 4.2). By definition $|C| = \ell$. Call s_A, s_B and s_C the number of edges induced by A, B and C. Further call s_{AB}, s_{BC} and s_{CA} the number of edges between A and B, B and C, and C and A.

A vertex $c \in C$ must share ℓ common neighbors with a, and b is only one of them. The other $\ell - 1$ common neighbors must lie either in C or in A. So for each vertex $c \in C$ there holds $d_C(c) + d_A(c) = \ell - 1$. Accordingly there must hold $d_C(c) + d_B(c) = \ell - 1$. Summing these equations over all $c \in C$ leads to

$$2s_C + s_{CA} = \ell(\ell - 1), \qquad 2s_C + s_{BC} = \ell(\ell - 1) \tag{4.1}$$

and hence also to $s_{CA} = s_{BC}$.

A vertex $a' \in A$ must share exactly ℓ common neighbors with b, and a is only one of them. The other $\ell - 1$ neighbors must lie either in B or in C and therefore there holds $d_B(a') + d_C(a') = \ell - 1$. Accordingly we derive $d_A(b') + d_C(b') = \ell - 1$ for all $b' \in B$. Summing over all $a' \in A$ (resp. $b' \in B$) yields

$$s_{AB} + s_{CA} = (\ell - 1)|A|, \qquad s_{AB} + s_{BC} = (\ell - 1)|B|. \tag{4.2}$$

By combining these equations in (4.2) we find

$$|A| - |B| = \frac{s_{CA} - s_{BC}}{\ell - 1}$$

and since we know $s_{CA} = s_{BC}$ and $\ell \geq 2$ we also found $|A| = |B|$ and therefore $d_G(a) = |A| + |C| + 1 = |B| + |C| + 1 = d_G(b)$. So all adjacent vertices are of the same degree and since G is connected G is regular. □

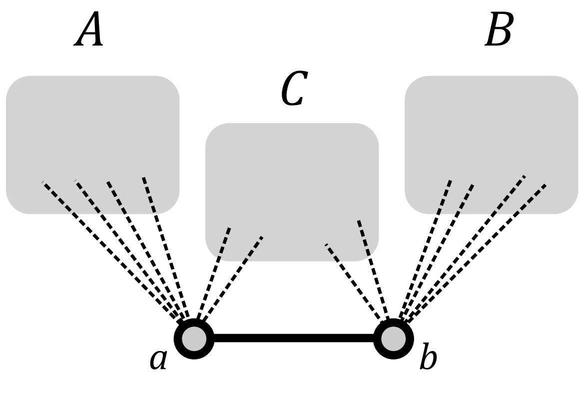

Figure 4.2: Configuration used in the proof of Theorem 4.3.

An alternative proof for the regularity can be found in [14]. Indeed, this result shows that $P_\ell(2)$-graphs for $\ell \geq 2$ are just a special case of the so called *strongly* *regular graphs.* A graph G is called *strongly regular* with the parameters n, d, λ and μ (or short an (n, d, λ, μ)-graph) if G is d-regular on n vertices and there holds

(i) any two adjacent vertices have exactly λ neighbors in common.

(ii) any two non-adjacent vertices have exactly μ neighbors in common.

The $P_\ell(2)$-graphs for $\ell \geq 2$ are just the (n, d, ℓ, ℓ)-graphs. The field of strongly regular graphs is well studied and many deep results are known. Nevertheless, we are going to prove the properties important for this topic only in the context of $P_\ell(2)$-graphs. We will show that a (regular) $P_\ell(2)$-graph is edge extremal without $K_{2,\ell+1}$-subgraph.

Lemma 4.4. *The number of vertices of a d-regular $P_\ell(2)$-graph G is*

$$n = \frac{d(d-1)}{\ell} + 1. \tag{4.3}$$

In particular, ℓ has to divide $d(d-1)$.

Proof. Count the number of paths of length two in G starting in a fixed vertex $v \in V$. For the first step there are d possibilities. For the second step there are $d-1$ because we are not allowed to go the same step back. So there is a total count of $d(d-1)$ such paths. Since there are exactly ℓ paths of length two from v to any other vertex, there must hold $\ell(n-1) = d(d-1)$. Solving this for n gives Eq. (4.3) $\square$

Lemma 4.5 (cf. [8])**.** *For $n, \ell \in \mathbb{N}$ the number of edges in an n-vertex graph G without $K_{2,\ell+1}$-subgraph is bounded from above by*

$$|E| \leq \frac{n}{4}\left(1 + \sqrt{4\ell(n-1)+1}\right). \tag{4.4}$$

Proof. Let G be a $K_{2,\ell+1}$-free graph on n vertices. We are counting the paths of length two. If there is a vertex v of degree d then one can choose any two of its neighbors to create a path of length two with v as the middle vertex. Thus there are exactly $\binom{d}{2}$ 2-paths with the middle vertex v (note that this also works for $d = 0$ and $d = 1$). Considering all vertices as possible middle vertices there is a total count of

$$\sum_{v \in V} \binom{d_G(v)}{2} = \frac{1}{2} \sum_{v \in V} d_G^2(v) - |E|$$

2-paths in G. Since there are no $K_{2,\ell+1}$ in G, between any two vertices there are at most ℓ paths of length two and therefore

$$\ell \binom{n}{2} \geq \frac{1}{2} \sum_{v \in V} d_G^2(v) - |E|. \tag{4.5}$$

Recall the discrete Cauchy-Schwarz-inequality $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$ and use $a_i = d_G(v_i)$ and $b_i = 1$ to find

$$\left(\sum_{v\in V} d_G(v)\right)^2 \leq n \sum_{v\in V} d_G^2(v).$$

Using this in (4.5) yields

$$\ell\binom{n}{2} \geq \frac{1}{2n}\left(\sum_{v\in V} d_G(v)\right)^2 - |E| = \frac{2}{n}|E|^2 - |E|.$$

This is a quadratic equation in $|E|$ with parameters n and ℓ. Solving this leads directly to Eq. (4.4). $\square$

Corollary 4.6. *A regular $P_\ell(2)$-graph is edge extremal without $K_{2,\ell+1}$-subgraph.*

Proof. We substitute Equation (4.3) into the upper bound (4.4):

$$\begin{aligned}
\mathrm{ex}(n, K_{2,\ell+1}) &\leq \frac{n}{4}\left(1 + \sqrt{4\ell(n-1)+1}\right) \\
&= \frac{n}{4}\left(1 + \sqrt{4d^2 - 4d + 1}\right) \\
&= \frac{n}{4}\left(1 + \sqrt{(2d-1)^2}\right) \\
&= \frac{nd}{2} = |E|.
\end{aligned}$$

$\square$

Unlike the windmills in the case $\ell = 1$ there are no examples of $P_\ell(2)$-graphs for $\ell \geq 2$ which are not extremal. With this result we found a sufficient and efficiently checkable condition for extremality. Unfortunately the following theorem shows that this is not too useful.

Theorem 4.7. *Let G be a d-regular $P_\ell(2)$-graph. Then $d - \ell$ divides ℓ^2.*

Proof. Consider the adjacency matrix A of G. The square of A must be of the form $A^2 = \ell\mathbb{1} + (d-\ell)\mathbb{I}$ (compare the proof of the Friendship Theorem 1.5). The eigenvalues of A^2 are

$$\lambda_1 = \ell n + d - \ell, \qquad \lambda_2 = d - \ell$$

where λ_1 is simple and λ_2 of multiplicity $n-1$. So the eigenvalues of A are $\pm\sqrt{\lambda_1}$ and $\pm\sqrt{\lambda_2}$ and must sum up to zero since the trace of A vanishes. Assume w.l.o.g.

that A has the positive eigenvalue $\sqrt{\lambda_1}$ and that $\sqrt{\lambda_2}$ is of multiplicity c. Then there holds

$$\sqrt{\lambda_1} + c\sqrt{\lambda_2} - (n-1-c)\sqrt{\lambda_2} = 0.$$

We can write this as

$$\sqrt{\frac{\lambda_1}{\lambda_2}} = n - 1 - 2c$$

and since the right hand side is integral, also the left hand side has to be of this form. So there is an $f \in \mathbb{N}$ with $\lambda_1 = f^2\lambda_2$. Note that

$$n = \frac{\lambda_1 - \lambda_2}{\ell}, \qquad d = \lambda_2 + \ell.$$

Substituting this into (4.3) gives $\lambda_1 = \lambda_2^2 + 2\lambda_2\ell + \ell^2$ and finally combination with $\lambda_1 = f^2\lambda_2$ leads to

$$f^2 = \lambda_2 + 2\ell + \frac{\ell^2}{\lambda_2}.$$

So the right hand side has to be integer and since $\lambda_2 = d - \ell$ we are done. □

Observation 4.8. For a given $\ell \in \mathbb{N}$ there are only finitely many divisors of ℓ^2. Thus there are only finitely many possible degrees d of G and since n is completely determined by d there can only be a finite number of (regular) $P_\ell(2)$-graphs for fixed ℓ. We can also give an upper bound on the number of vertices for a (regular) $P_\ell(2)$-graph. Of course any divisor of ℓ^2 is at most ℓ^2 and hence $d \leq \ell^2 + \ell = \ell(\ell+1)$ which (using Lemma 4.4) leads to

$$\begin{aligned} n &= \frac{d(d-1)}{\ell} + 1 \leq \frac{\ell(\ell+1)(\ell(\ell+1)-1)}{\ell} + 1 \\ &= (\ell+1)(\ell(\ell+1)-1) + 1 \\ &= \ell(\ell+1)^2 - \ell - 1 + 1 \\ &= \ell((\ell+1)^2 - 1) \\ &= \ell(\ell^2 + 2\ell + 1 - 1) \\ &= \ell^2(\ell+2). \end{aligned}$$

In this sense our criterion is pretty weak because only a finite number of infinitely many extremal graphs are $P_\ell(2)$-graphs. At least these results allow to directly identify a couple of graphs as extremal graphs (cf. [14]):

- The complete graph $K_{\ell+2}$ is a $P_\ell(2)$-graph of degree $d = \ell + 1$ (and indeed $d - \ell = 1$ divides ℓ^2) and $K_{2,\ell+1}$-free.
- The *Shrikhande graph* and the $K_4 \times K_4$ are the only two $P_2(2)$-graphs, both 6-regular on 16 vertices and hence extremal $K_{2,3}$-free (see Fig. 4.3).

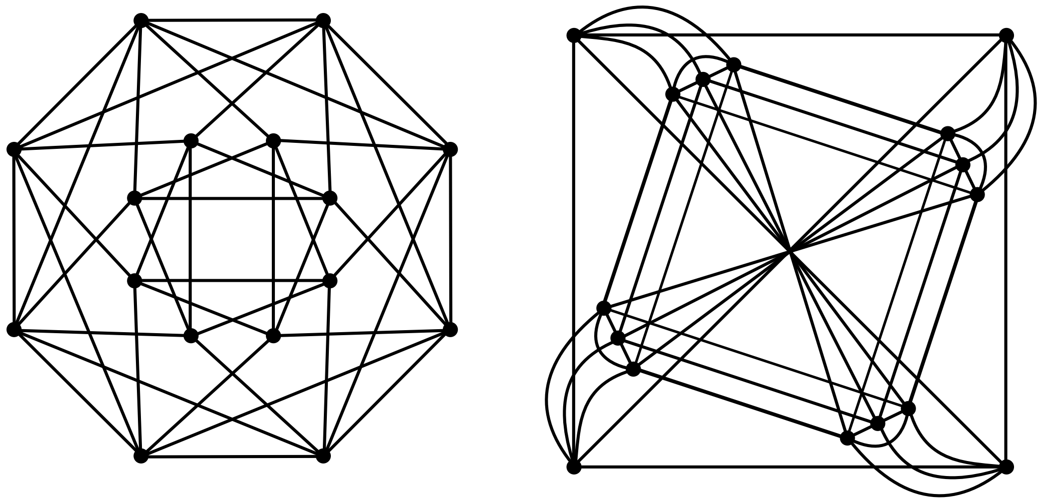

Figure 4.3: The Shrikhande graph (left) and the $K_4 \times K_4$ (right).

- The line graph $L(K_6)$ of the complete graph on 6 vertices is 8-regular and a $P_4(2)$-graph. It is extremal without a $K_{2,5}$-subgraph.
- The *halved* 5-*cube* $\frac{1}{2}Q_5$ on 16 vertices is 10-regular and a $P_6(2)$-graph. The graph $\frac{1}{2}Q_5$ is defined as a single connected component of the graph emerging from the vertices of the 5-cube by connecting vertices at distance two in Q_5. It is extremal $K_{2,7}$-free.

4.2 Infinite graphs

There is an easy way to construct an infinite $P_\ell(2)$-graph. Start with a $K_{2,\ell+1}$-free graph G_0. The next graph G_{i+1} emerges from G_i by the following operations:

1. For any two vertices $v, w \in V(G_i)$ with exactly $\ell' < \ell$ common neighbors, add $\ell - \ell'$ vertices $u^j_{vw}, j = 1, ..., \ell - \ell'$.
2. Join all these vertices u^j_{vw} to their corresponding vertices v and w. This adds $\ell - \ell'$ new common neighbors to the pair v and w.

The union $G := \bigcup_{i \in \mathbb{N}} G_i$ is a $P_\ell(2)$-graph. This is not obvious:

Claim 4.9. *The graph G constructed above is a $P_\ell(2)$-graph.*

Proof. We have to check that any two vertices $v, w \in V$ neither got more nor fewer than ℓ common neighbors.

Case $|N_G(v) \cap N_G(w)| < \ell$*:* Since $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ there holds $v, w \in V(G_i)$ for some $i \in \mathbb{N}_0$. So they have ℓ common neighbors in G_{i+1}, hence at least ℓ common neighbors in G. Contradiction.

Case $|N_G(v) \cap N_G(w)| > \ell$*:* Again there is a G_i where this already holds. We use induction on the construction step i to show that this is not possible. This is clear for G_0 by definition. So assume G_i is $K_{2,\ell+1}$-free and G_{i+1} is not. For the vertices $v, w \in V(G_{i+1})$ with more than ℓ common neighbors there are three possible configurations: both vertices were already contained in G_i, only w was in G_i or both vertices are new in G_{i+1}. Observe that the vertices u_{ab}^j added to G_i to construct G_{i+1} are *only* common neighbors of $a, b \in V(G_i)$. Since the construction adds exactly the right amount of common neighbors to any existing pair $v, w \in V(G_i)$ to achieve exactly ℓ of them, the vertices v and w cannot be both in G_i. If, e.g. v is a new added vertex, then v only has two neighbors and this can only be too much if $\ell = 1$. So it suffices to consider this case. If $w \in V(G_i)$ and w and v share the two vertices a and b in G_{i+1} then w is already a common neighbor of a and b in G_i. So no v were be added at all. Otherwise for $\ell = 1$ there will at most be added a single vertex for each pair $a, b \in V(G_i)$. So it could never have happened that v and w are both new to G_{i+1} and we are done. □

At this point it is not clear if G is infinite. This can be achieved by a clever choice of G_0. For example for $\ell = 1$ choose $G_0 = C_5$ (see Fig. 4.4). If G is finite the Friendship Theorem 1.5 requires G to be a windmill which, of course, do not contain a C_5. This also shows that the statement of the Friendship Theorem does not hold when including infinite graphs because G is a $P(2)$-graph but no windmill. For the other values of ℓ just choose G_0 to be a graph on more than $\ell^2(\ell+2)$ isolated vertices (i.e. G_0 contains no edges). As we know from Observation 4.8 these are too many vertices for a finite $P_\ell(2)$-graph. Another working approach to infinite $P(2)$-graphs can be found in [1] and [4]. There it was also proven that there are $2^{\aleph_\alpha}$ distinct $P(2)$-graphs for each infinite cardinal $\aleph_\alpha$.

A similar construction does not work for $P(k)$-graphs with $k \geq 3$ in the sense, that it cannot build infinite graphs. Consider the following construction based on the one above: Start with a graph G_0 without any double k-path. G_{i+1} emerges from G_i by adding a k-path consisting of new vertices between any pair of vertices that are not already joined by a k-path in G_i. We show that the infinite union $G := \bigcup_{i \in \mathbb{N}} G_i$ is either a finite graph or contains a double k-path. At first, observe that Observation 2.1, Lemma 2.4 and Observation 2.10 also work for infinite graphs. We conclude that G contains an even cycle C regardless of whether G is finite or not. This even cycle is already contained in a G_i for some $i \in \mathbb{N}_0$. If we assume that G is not finite, the sequence of constructed graphs cannot become stationary. Hence we have to add a k-path to G_i to achieve G_{i+1}. But in G_{i+1} this k-path either starts in C and does not intersect it again or is a k-chord of C. As we mentioned in

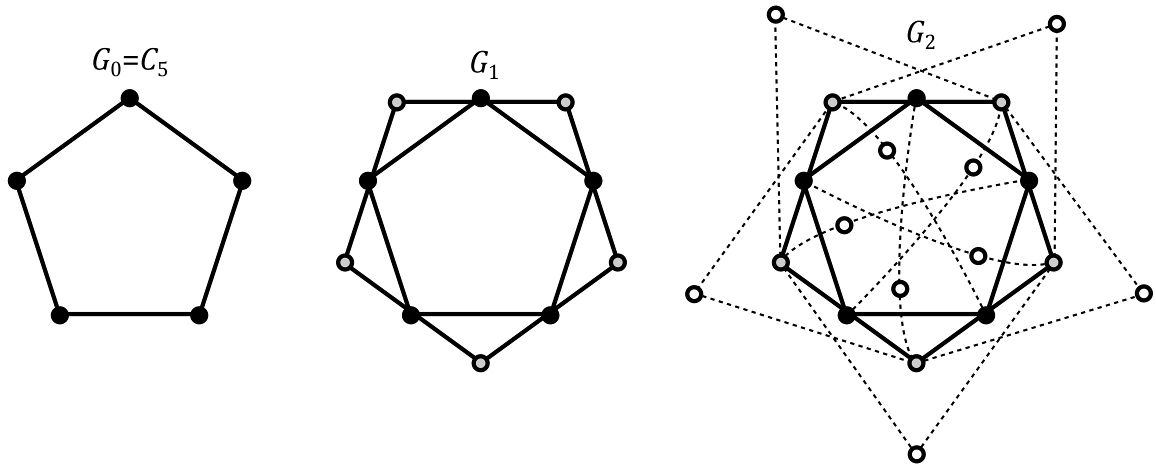

Figure 4.4: The first three steps of a construction of an infinite $P(2)$-graph which is not a windmill.

Observation 2.1 and 2.10 this is not possible in a $P(k)$-graph.

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Chemnitz, den

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