Spectral Realizations of Symmetric Graphs, Spectral Polytopes and Edge-Transitivity

Between Spectral Graph Theory and Polytope Theory

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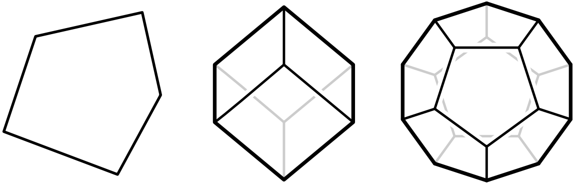
TECHNISCHE UNIVERSITÄT CHEMNITZ

An Overview

Convex polytopes



Definition.

A **(convex) polytope** is the convex hull of finitely many points in $\mathbb{R}^d$.

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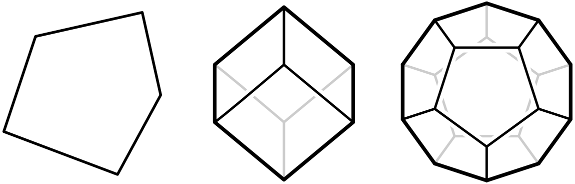

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Polytopes have

- vertices,
- edges,
- faces.

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ intersection of P with a hyperplane

The edge-graph and the quest for reconstruction

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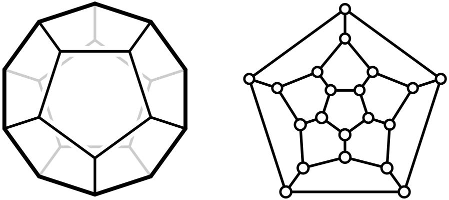
The **edge-graph** of P is the graph $G_P = (V, E)$ with

- $V := \{1, ..., n\}$. (corresponding to the vertices $v_1, ..., v_n$ of P)
- $i, j \in V$ are adjacent in $G_P \iff \mathrm{conv}\{v_i, v_j\}$ is an edge of P



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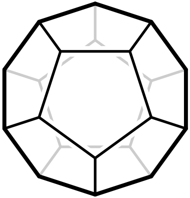
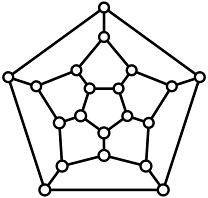
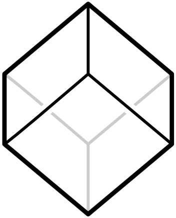




Note: the edge-graph of P carries very little information about the polytope.

A curious (spectral) observation



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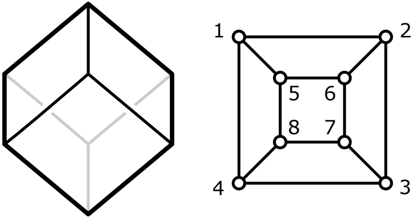

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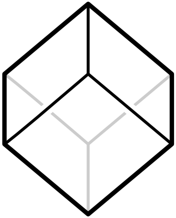

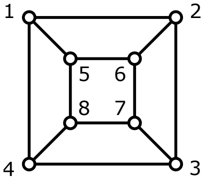

$$
\begin{array}{cc}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} & \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
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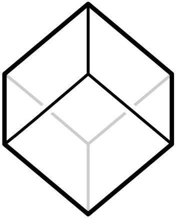

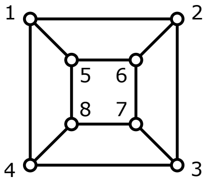

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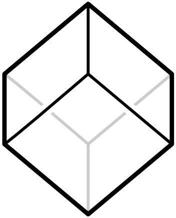

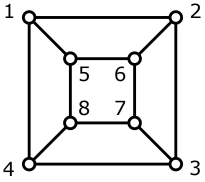

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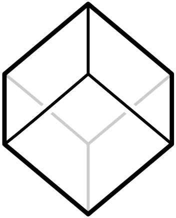

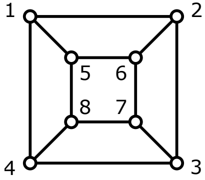

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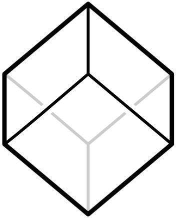

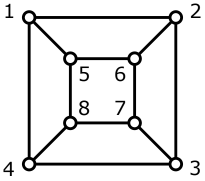

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θ_2-eigenvectors of G_P $\longrightarrow$ θ_2-spectral realization v^θ of G_P

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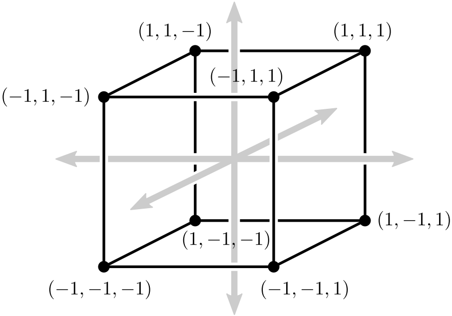

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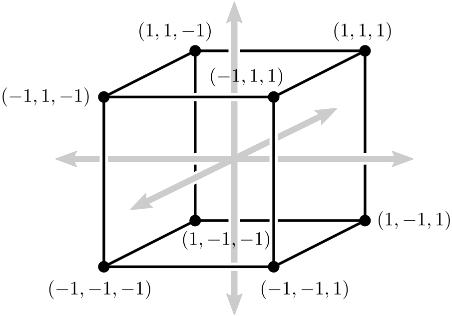

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- For which polytopes does this work?
- Why have we used θ_2?

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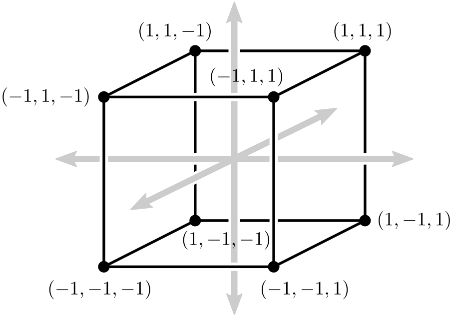

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$\longrightarrow$ **Part I**

Eigenpolytopes and spectral polytopes

Definition. (Godsil, 1978)

The **eigenpolytope** of a graph G to eigenvalue $\theta \in \mathrm{Spec}(A(G))$ is

$$P_G(\theta) := \mathrm{conv}\{v_1^\theta, ..., v_n^\theta\}.$$

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- Licata & Powers, *A Surprising Property of some Regular Polytopes*, 1978.
- Mohri, *The θ_1-Eigenpolytopes of the Hamming Graphs*, 1997.
- Godsil, *Eigenpolytopes of Distance Regular Graphs*, 1998.

Properties of spectral polytopes

Observations

If P is θ-spectral, then

- rigidity: P is uniquely determined by its edge-graph.
- symmetry: P is as symmetric as its edge-graph.

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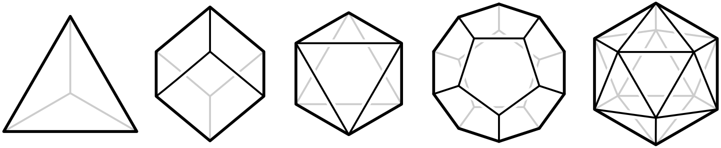
Regular polytopes are θ_2-spectral: (LICATA & POWERS, 1986)



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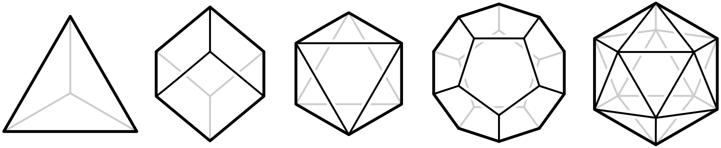
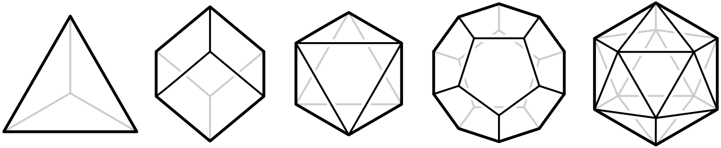


Maybe every <u>sufficiently symmetric</u> polytope is spectral.

What is sufficient symmetry?

Regularity is sufficient: (Licata & Powers, 1986)



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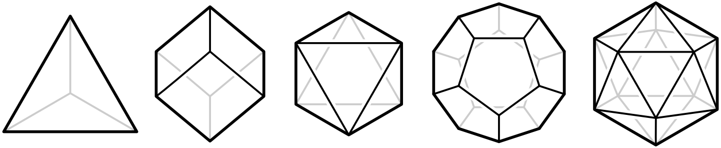
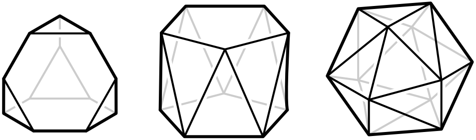


Vertex-transitivity is <u>not</u> sufficient: $\mathrm{Aut}(P) := \{T \in \mathrm{O}(\mathbb{R}^d) \mid TP = T\}$



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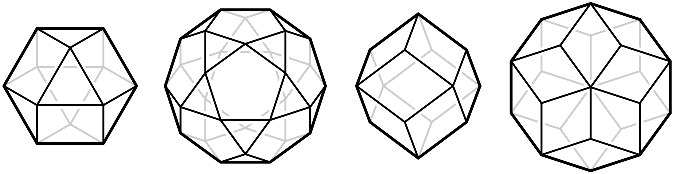

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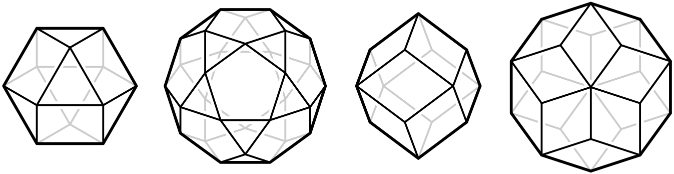

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- Can they be classified?

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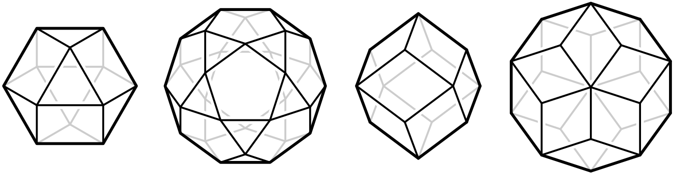

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$\longrightarrow$ **Part II**

Papers

Published

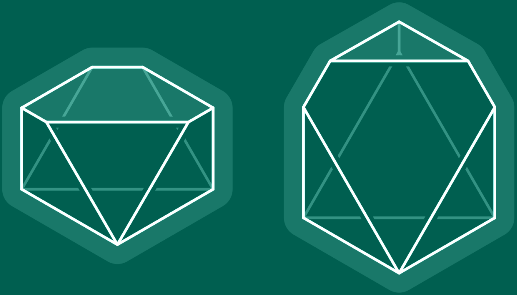
- *Vertex-Facets Assignments for Polytopes* (with Thomas Jahn)
  Contributions to Algebra and Geometry
- *Geometry and Topology of Symmetric Point Arrangements*
  Linear Algebra and its Applications
- *The Classification of Vertex-Transitive Zonotopes*
  Discrete & Computational Geometry

Manuscripts

- *The Edge-Transitive Polytopes that are not Vertex-Transitive*
- *Symmetric and Spectral Realizations of Highly Symmetric Graphs*
- *Eigenpolytopes, Spectral Polytopes and Edge-Transitivity*

Part I

Spectrum and Symmetry



The Izmestiev construction

(Izmestiev, 2008)

$$P^\circ := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le 1 \text{ for all } i \in V\}.$$

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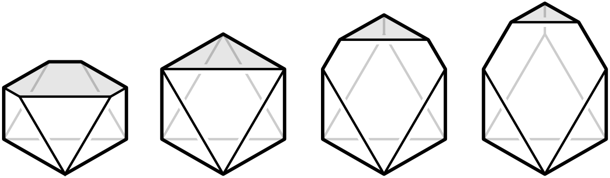

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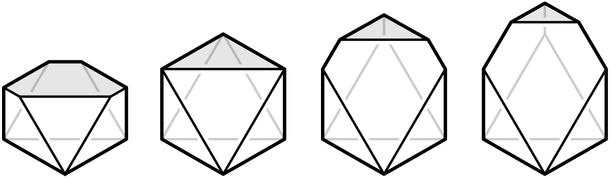

Definition.

The **Izmestiev matrix** is $M \in \mathbb{R}^{n\times n}$ with

$$M_{ij} := \frac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}.$$

A sufficient criterion for spectral polytopes

Theorem.

Let M be the Izmestiev matrix of P. If

(i) M_{ii} *is the same for all* $i \in \{1, ..., n\}$ *and*

(ii) M_{ij} *is the same for all edges* $ij \in E(G_P)$,

then P *is* θ_2*-spectral.*

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$$M_{ij} := \frac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\,\partial c_j}\Big|_{c=(1,...,1)} \overset{\text{if } ij \in E}{=} -\frac{\overbrace{\mathrm{vol}(f_{ij}^\circ)}^{\text{dual face to edge } ij}}{\|v_i\|\|v_j\| \sin\sphericalangle(v_i, v_j)}.$$

A sufficient criterion for spectral polytopes

Theorem.

Let M be the Izmestiev matrix of P. If

(i) M_{ii} *is the same for all* $i \in \{1, ..., n\}$ *and*

(ii) M_{ij} *is the same for all edges* $ij \in E(G_P)$,

$\iff M = \alpha \,\mathrm{Id} + \beta A$

then P *is* θ_2*-spectral.*

$$M_{ij} := \frac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i \, \partial c_j}\Big|_{c=(1,\dots,1)} \overset{\text{if } ij \in E}{=} -\frac{\overset{\text{dual face to edge } ij}{\mathrm{vol}(f_{ij}^\circ)}}{\|v_i\|\|v_j\| \sin \sphericalangle(v_i, v_j)}.$$

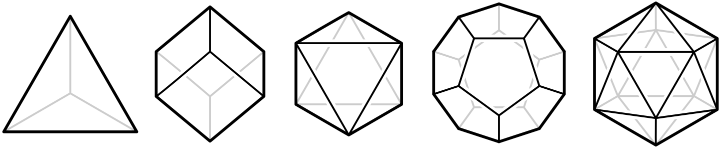
Corollary.

If P *is simultaneously vertex- and edge-transitive, then* P *is* θ_2*-spectral.*

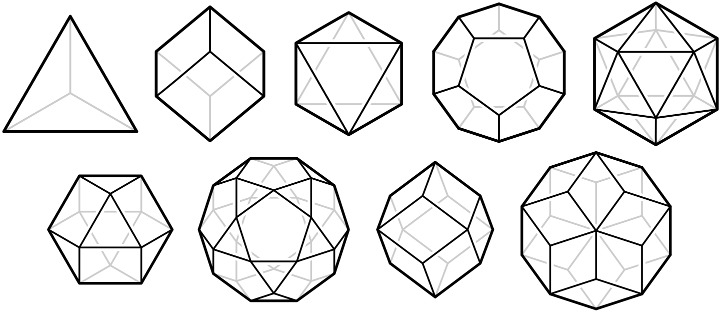
Part II

Edge-Transitive Polytopes

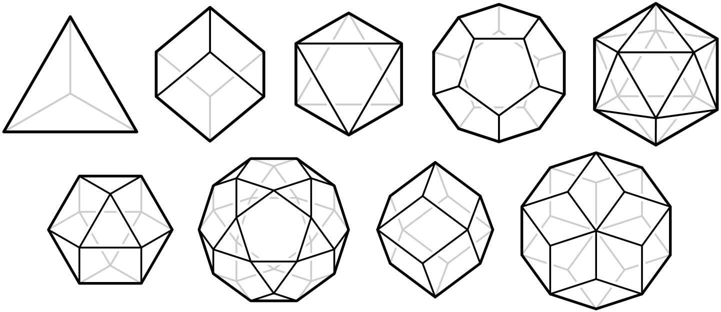
Edge-transitive polyhedra



Edge-transitive polyhedra



Edge-transitive polyhedra



Theorem. (Grünbaum & Shephard, 1987)

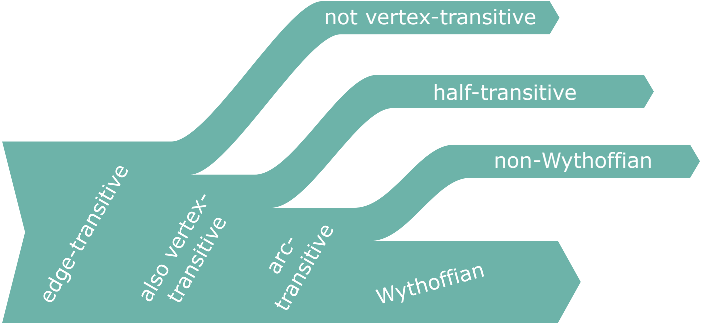
There are nine edge-transitive polyhedra.

Transitivity in polytopes

No serious consideration seems to have been given to polytopes in dimensions $d \geq 4$ about which transitivity of the symmetry group is assumed only for faces of suitably low dimensions, and regularity or some variant of it is required only for faces of dimensions $\leq d-2$.

– GRÜNBAUM (*Convex Polytopes*, 1967/2003)

A hierarchy of edge-transitive polytopes



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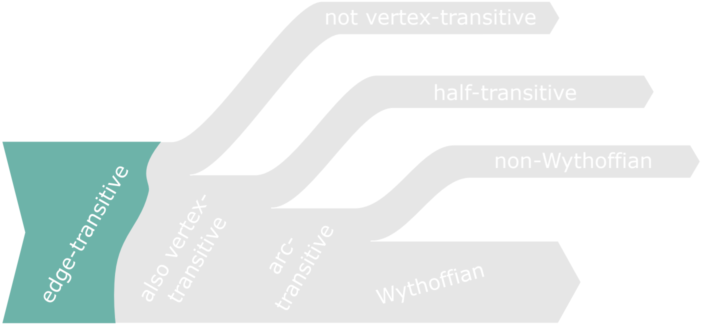

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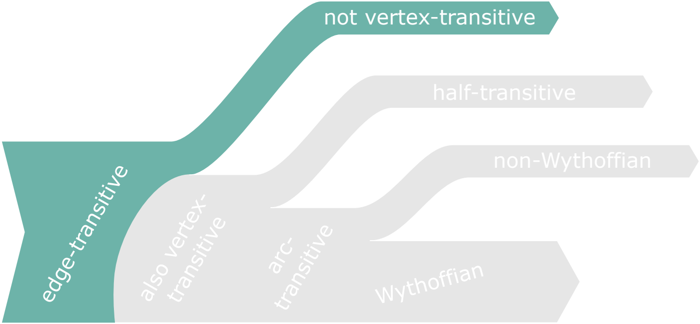
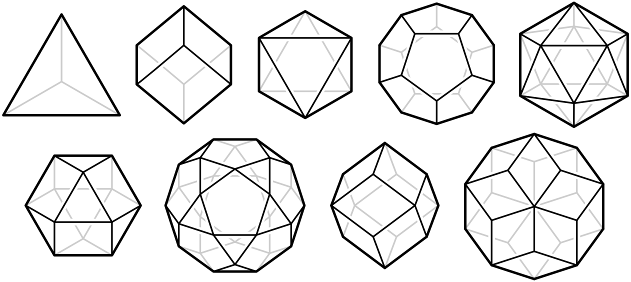


Edge- but not vertex-transitive



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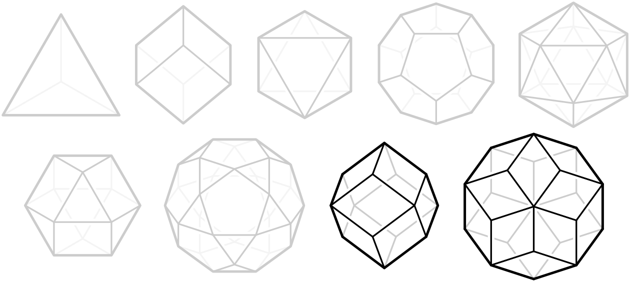

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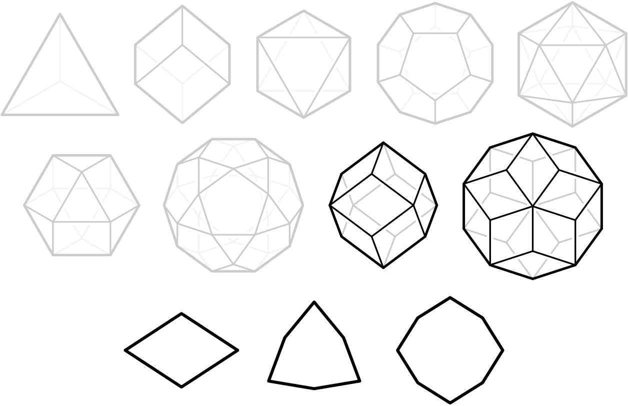

Edge-transitive but not vertex-transitive

Theorem.

An edge-transitive polytope in dimension $d \geq 4$ is vertex-transitive.

Edge-transitive but not vertex-transitive

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An edge-transitive polytope in dimension $d \geq 4$ is vertex-transitive.

Proof.

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Proof.

If P is edge-transitive but not vertex-transitive, then

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If P is edge-transitive but not vertex-transitive, then

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(ii) P has an *edge-insphere*, and

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If P is edge-transitive but not vertex-transitive, then

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(iii) the edge-graph G_P is *bipartite*.

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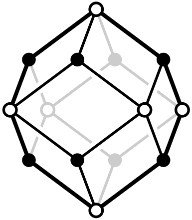
Proof.



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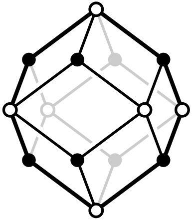

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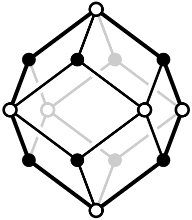

Theorem.

If P is bipartite and of dimension $d \geq 4$ then it is a Γ-permutahedron,

Edge-transitive but not vertex-transitive

Theorem.

An edge-transitive polytope in dimension $d \geq 4$ is vertex-transitive.

Proof.

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P is **bipartite** if

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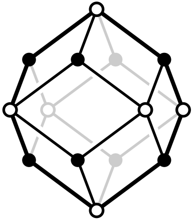
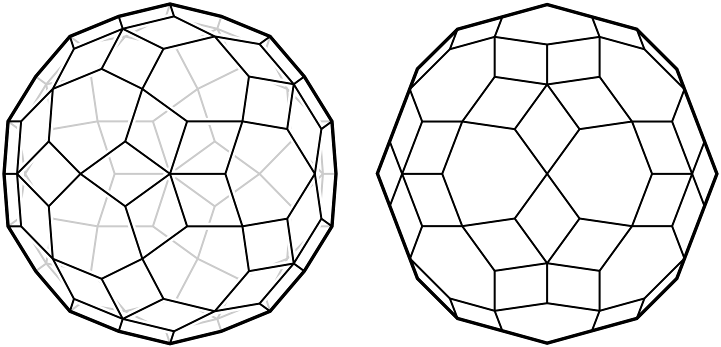


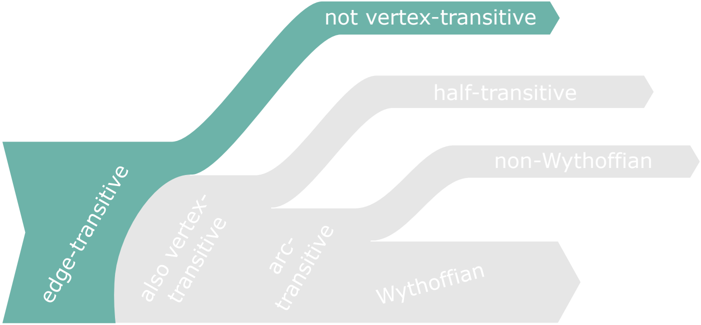
Theorem.

If P is bipartite and of dimension $d \geq 4$ then it is a Γ-permutahedron, and therefore vertex-transitive.

An almost bipartite polyhedron



A hierarchy of edge-transitive polytopes



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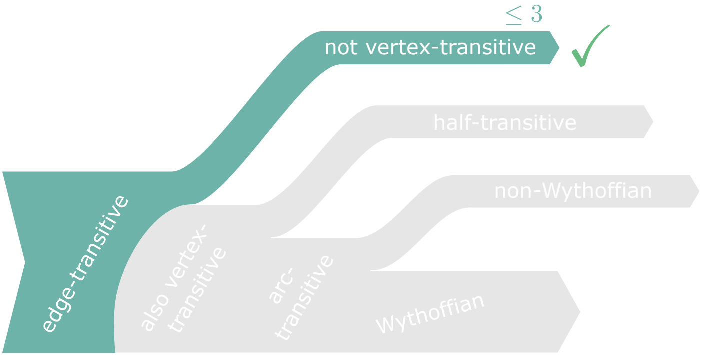

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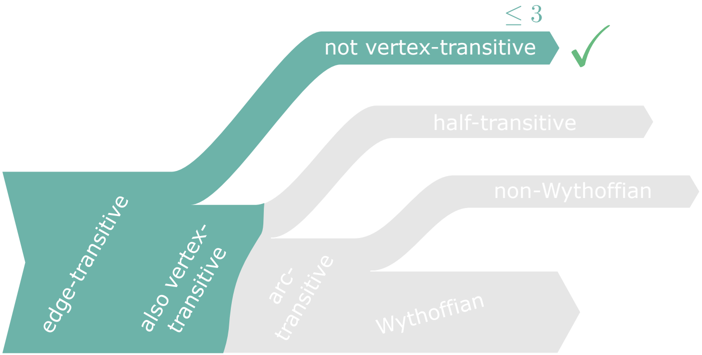


Vertex- and edge-transitive polytopes

Theorem.

If P is both vertex- and edge-transitive, then

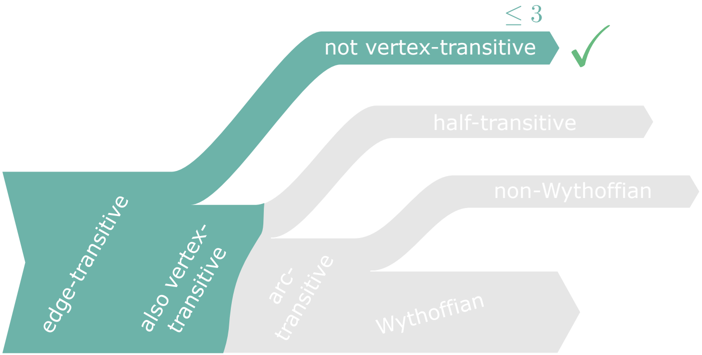
- P *is* θ_2*-spectral.*
- P *is uniquely determined by its edge-graph (up to scale and orientation).*
- P *is as symmetric as its edge-graph.*
- $\mathrm{Aut}(P)$ *is irreducible.* ($\mathrm{Aut}(P)$ *fixes no non-trivial subspace)*
- P *has edge-length* ℓ *and circumradius* r *with*

$$\frac{\ell}{r} = \sqrt{2 - \frac{2\theta_2}{\deg(G_P)}}.$$

- *the polar dual* P° *has dihedral angle* α *with*

$$\cos(\alpha) = -\frac{\theta_2}{\deg(G_P}.$$

A hierarchy of edge-transitive polytopes



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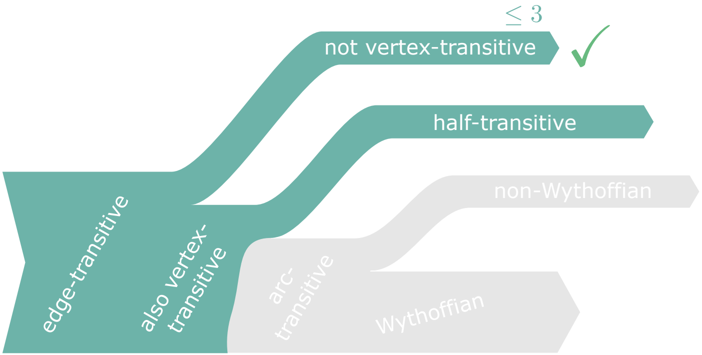
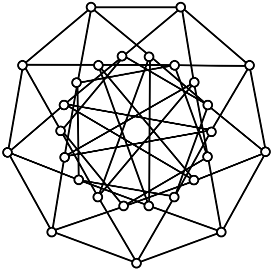


Half-transitive polytopes

Corollary.

The edge-graph of a half-transitive polytope must be itself half-transitive.



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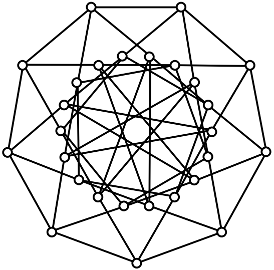
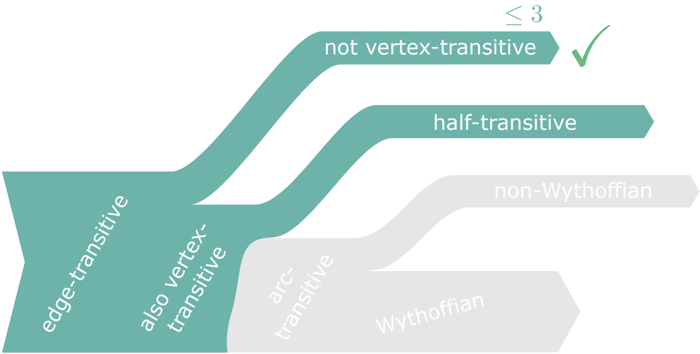


Conjecture.

There are no half-transitive polytopes.

A hierarchy of edge-transitive polytopes



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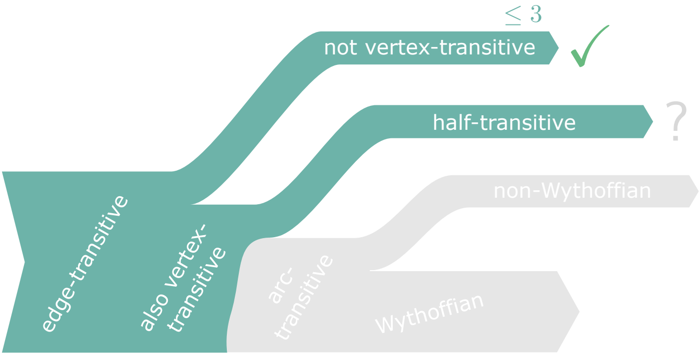

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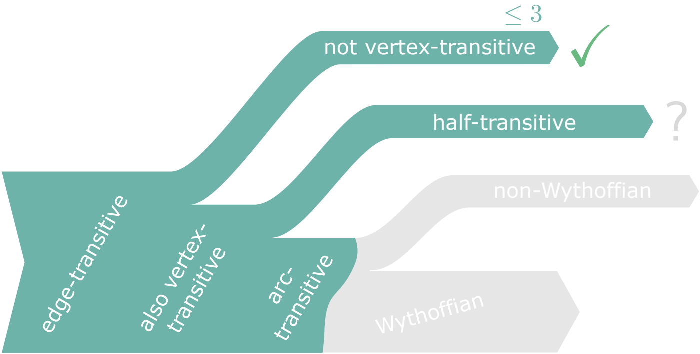

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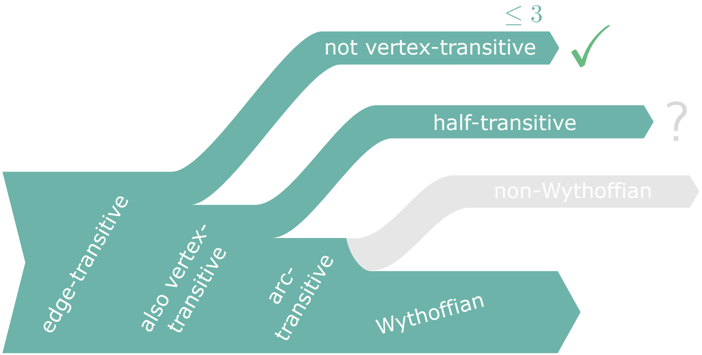

Wythoffian polytopes

Definition.

Let $\Gamma \subseteq \mathrm{GL}(\mathbb{R}^d)$ be a matrix group and $x \in \mathbb{R}^d$. The **orbit polytope** $\mathrm{Orb}(\Gamma, x)$ is

$$\mathrm{Orb}(\Gamma, x) := \mathrm{conv}\{Tx \mid T \in \Gamma\}.$$

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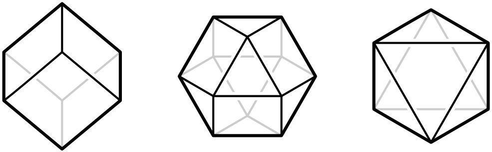

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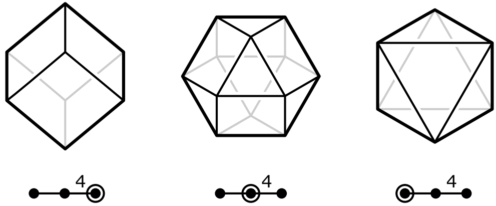

Wythoffian arc-transitive polytopes

Definition.

A Coxeter-Dynkin diagram is called **transitive** if its symmetry group acts transitively on the ringed nodes.

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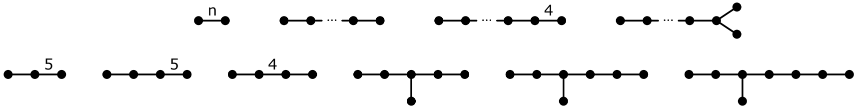

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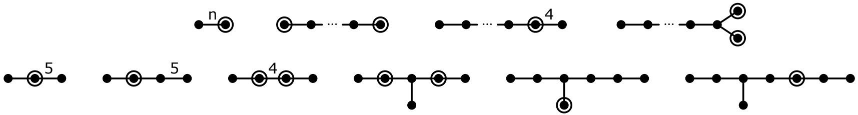

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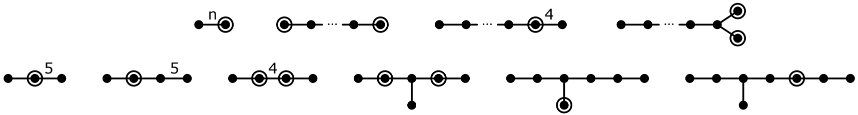

Conjecture.

A Wythoffian polytope is arc-transitive if and only if its Coxeter-Dynkin diagrams is transitive.

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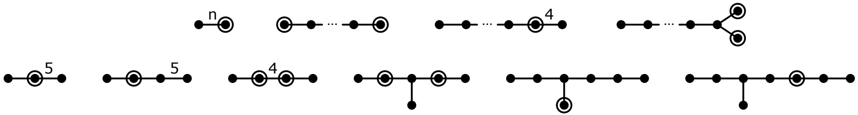

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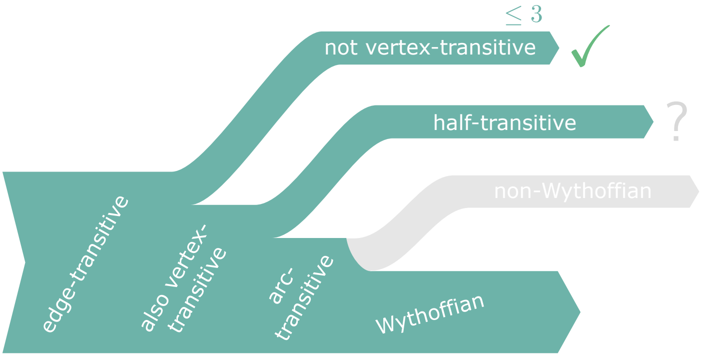
d	1	2	3	4	5	6	7	8	≥ 9
$\#$	1	∞	7	15	11	19	22	25	$2d+1$

A hierarchy of edge-transitive polytopes



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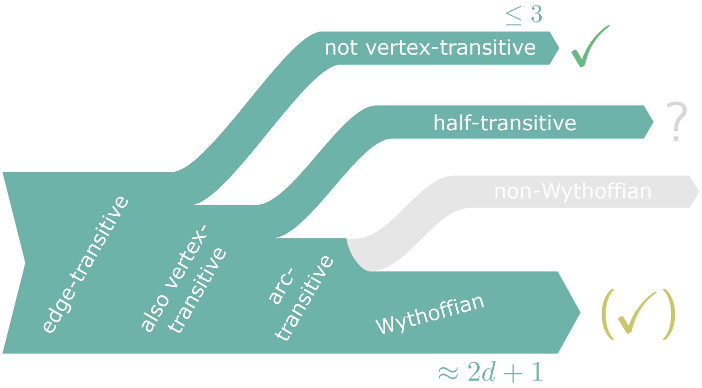

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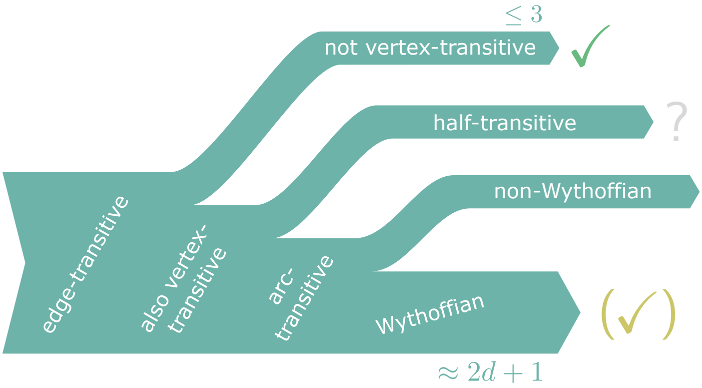

Non-Wythoffian polytopes

Wythoffian polytope = orbit polytope of reflection group

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Definition.

- $T \in \mathrm{O}(\mathbb{R}^d)$ is a **reflection** if $\mathrm{Spec}(T) = \{(-1)^1, 1^{d-1}\}$.
- A **reflection group** is a matrix group generated by reflections.

Non-Wythoffian polytopes

Wythoffian polytope = orbit polytope of reflection group

Definition.

- $T \in \mathrm{O}(\mathbb{R}^d)$ is a **k-reflection** if $\mathrm{Spec}(T) = \{(-1)^k, 1^{d-k}\}$.
- A **k-reflection group** is a matrix group generated by k-reflections.

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Theorem.

Every arc-transitive polytope is an orbit polytope of a k-reflection group.

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Caveat: k-reflection groups are not well understood and rather general.

Non-Wythoffian polytopes

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Every arc-transitive polytope is an orbit polytope of a k-reflection group.

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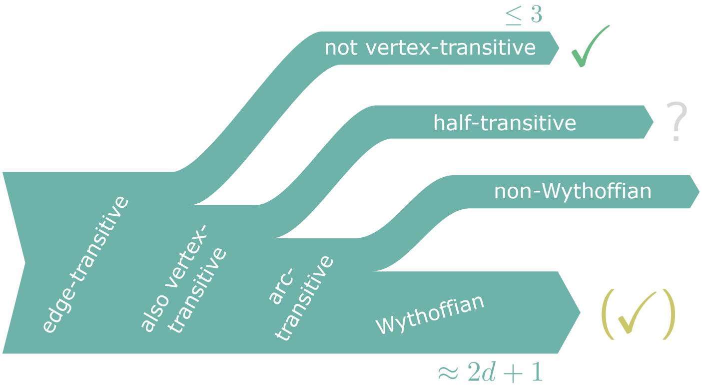
Conjecture.

All arc-transitive polytopes are orbit polytopes to 1-reflection groups.

A hierarchy of edge-transitive polytopes



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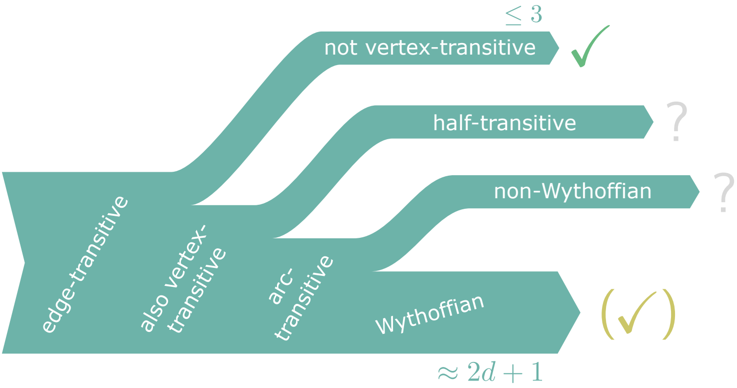
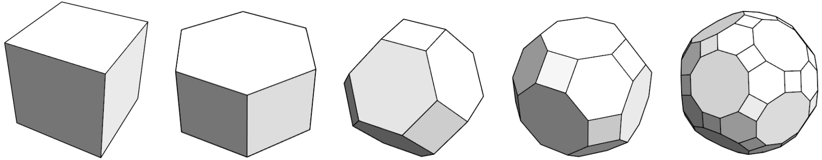


Other Results

Classification of vertex-transitive zonotopes

Definition.

A **zonotope** is a polytope with only centrally symmetric faces.



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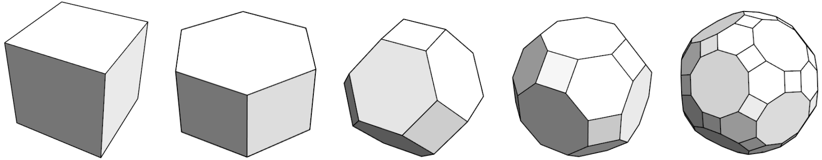


Theorem.

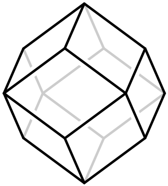
If Z is a zonotope that is either

(i) *vertex-transitive, or*

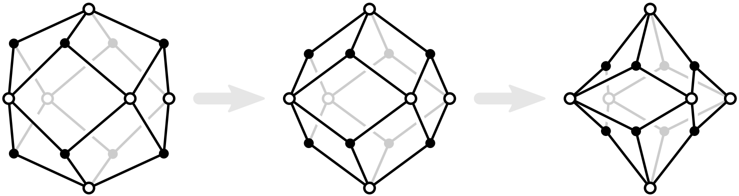
(ii) *inscribed with all edges of the same length,*

then Z is a Γ-permutahedron. *(a generic orbit polytope of the reflection group Γ)*

Rigidity of graph realizations



Rigidity of graph realizations



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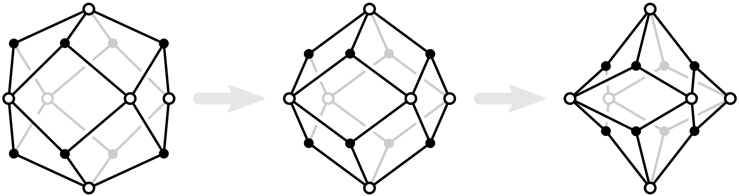


Theorem.

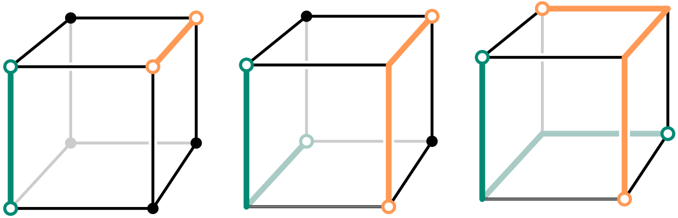
If v is a **distance-transitive** *(and irreducible) graph realization, then*

- *v is a spectral realization*
- *v is rigid*
- *v is as symmetric as the graph.*

Distance-transitivity

Definition.

A graph G is **distance-transitive** if for any two pair of vertices $i, j, \hat{\imath}, \hat{\jmath} \in V$ with $\text{dist}(i, j) = \text{dist}(\hat{\imath}, \hat{\jmath})$ exists a $\sigma \in \text{Aut}(G)$ with $\sigma(i) = \hat{\imath}$ and $\sigma(j) = \hat{\jmath}$.



Distance-transitive polytopes

Theorem. (based on a classification by Godsil, 1997)

If $P \subset \mathbb{R}^d$ is distance-transitive, then P is one of the following:

- *a regular polygon,*
- *the icosahedron,*
- *the dodecahedron,*
- *a crosspolytope,*
- *a hyper-simplex (this includes regular simplices),*
- *a demi-cube,*
- *a cartesian power of a simplex (this includes hypercubes),*
- *the 6-dimensional 2_{21}-polytope,*
- *the 7-dimensional 3_{21}-polytope.*

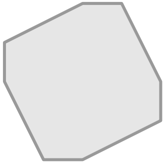
Outlook

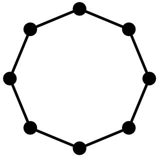
Many open questions

Questions

- Is the Izmestiev criterion characterizing spectral polytopes?
- Are there half-transitive or non-Wythoffian arc-transitive polytopes?
- Is my conjectured classification of edge-transitive polytopes complete?
- Can we classify k-face transitive polytopes?
- What are the inscribed zonotopes?
- ...

Capturing symmetries via colors





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There is a coloring $\mathfrak{c} \colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ *of the edge-graph so that*

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{GL}}(P).$$

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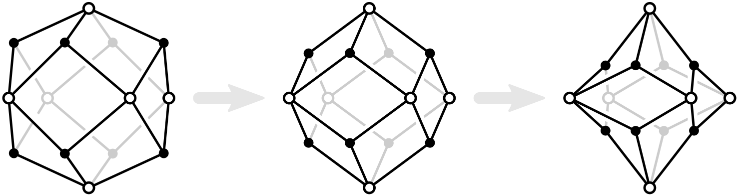
Idea: use $\mathfrak{c}(i) = M_{ii}$ and $\mathfrak{c}(ij) = M_{ij}$. (where M is the Izmestiev matrix)

Algebraic criteria for symmetric rigidity



Question

Can a graph realization (or an arrangement of points) be deformed without loosing a prescribed set of symmetries $\Sigma \subseteq \mathrm{Sym}(V)$?

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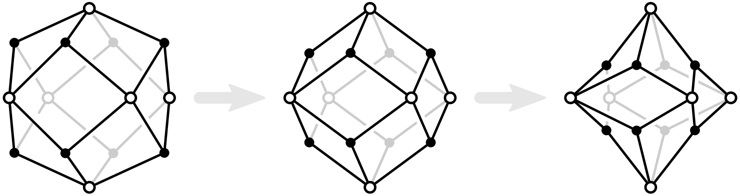

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Theorem.

An arrangement is Σ-rigid if and only if its Bose-Mesner algebra is commutative.

Thank you.