A SHORT POLYHEDRAL PROOF OF THE INFINITESIMAL STOKER CONJECTURE

MARTIN WINTER[†]

Abstract. The infinitesimal Stoker conjecture states that an infinitesimal deformation of a convex polytope that preserves all dihedral angles also preserves all normal directions up to isometry. We give a short polyhedral proof, self-contained up to assuming spectral and geometric properties of the Izmestiev matrix.

1. Introduction

In 1968 Stoker asked whether the dihedral angles of a convex Euclidean polyhedron determine its face angles [9]. A first infinitesimal analogue was formulated by Schlenker [7], which subsequently became known as the *infinitesimal Stoker conjecture*. In this note we work with the following formulation:

Theorem 1. *Let $P \subset \mathbb{R}^d, d \geq 2$ be a convex polytope with facets F_i, outward unit normal vectors n_i and dihedral angles θ_{ij}. Given an infinitesimal deformation of P with $\dot{\theta}_{ij} \geq 0$ whenever F_i and F_j are adjacent, then there exists a skew-symmetric matrix $S \in \mathbb{R}^{d\times d}$ with $\dot{n}_i = Sn_i$ for all i.*

A first proof of the infinitesimal Stoker conjecture by Mazzeo & Montcouquiol appeared in 2011 [5] and established the conjecture in dimension three using deformation theory of cone-manifolds. Another approach in dimension three, though not explicitly stated as such, can be extracted from the works of Weiss [10,11]. Adiprasito observed the arbitrary-dimensional infinitesimal statement and sketched an argument via Hodge–Riemann relations [1].

In this note we give the first purely polyhedral proof of the infinitesimal Stoker conjecture, and the first direct proof in Euclidean space of arbitrary dimension. The proof was initially based on the prestress stability of coned polytope frameworks established in [6]. It was subsequently distilled to be self-contained modulo known spectral and geometric properties of the *Izmestiev matrix* [4]. The only ingredients left hidden within [4] are Minkowski's second inequality as well as Bol's condition for equality. For the interested reader, Section 4 contains a few comments on how this proof came about, as well as pointers to literature that develops the relevant ideas in greater detail.

[†]Max-Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

Email address: martin.winter@mis.mpg.de.

Date: September 15, 2026.

2020 *Mathematics Subject Classification.* 51M20, 52B11, 52C25, 52A39, 15A63.

Key words and phrases. infinitesimal Stoker conjecture, convex polytopes, infinitesimal rigidity, dihedral angles, Izmestiev matrix, Lorentzian quadratic form.

2. Setup and assumptions

All dotted variables like $\dot n_i$ and $\dot\theta_{ij}$ denote formal first-order variations, not derivatives. A rigorous treatment can be implemented via dual numbers $\mathbb{R}[\epsilon]/\epsilon^2$. We suppress such details for readability. This section lists the implicit assumptions made in the statement of Theorem 1.

The n_i stay normalized throughout the deformation, hence

$$\langle n_i, \dot n_i\rangle = 0, \quad \text{for all facets } F_i. \tag{1}$$

As observed by Karim Adiprasito in [1], the one-sided condition $\dot\theta_{ij} \ge 0$ is easily reduced to its equality version: the classical Schläfli formula states

$$\sum_{i\neq j} V_{ij}\dot\theta_{ij} = 0,$$

where V_{ij} is the volume of the ridge $R_{ij} = F_i \cap F_j$. Thus, assuming $\dot\theta_{ij} \ge 0$ and by $V_{ij} > 0$, we necessarily have $\dot\theta_{ij} = 0$. In the following we will directly assume the equality variant. Since $\theta_{ij} = \pi - \arccos\langle n_i, n_j\rangle$, we can do so by imposing

$$\langle n_i, \dot n_j\rangle + \langle \dot n_i, n_j\rangle = 0, \quad \text{whenever } i \sim j, \tag{2}$$

where $i \sim j$ means that the facets F_i and F_j are adjacent.

Lastly, the deformations considered preserve the combinatorial type of the polytope. We make use of this by assuming that Minkowski balancing identities on the facets stay valid under variation: for each i

$$\sum_{j:j\neq i} V_{ij} n_{ij} = 0 \quad\implies\quad \sum_{j:j\neq i} \dot V_{ij} n_{ij} + \sum_{j:j\neq i} V_{ij}\dot n_{ij} = 0, \tag{3}$$

where n_{ij} is the normal vector of the ridge R_{ij} considered as a facet of F_i. We have

$$n_{ij} := \frac{n_j + \cos(\theta_{ij})n_i}{\sin(\theta_{ij})} = \frac{n_j - \langle n_i, n_j\rangle n_i}{\sin(\theta_{ij})}, \tag{4}$$

and since $\dot\theta_{ij} = 0$, the variation can be expressed as

$$\dot n_{ij} = \frac{\dot n_j + \cos(\theta_{ij})\dot n_i}{\sin(\theta_{ij})} = \frac{\dot n_j - \langle n_i, n_j\rangle \dot n_i}{\sin(\theta_{ij})}. \tag{5}$$

3. Proof of Theorem 1

Let h_i be the height of the facet $F_i \subset P$ over the origin. For convenience we may assume $0 \in \mathrm{int}(P)$, then $h_i > 0$, though strictly speaking this is not used.

The presented proof is in large parts a study of the quadratic form defined by the Alexandrov matrix.[1] The latter is the only major external ingredient to the proof that we need to assume and that we recall now.

---

[1] In [6] this corresponds to the energy of the Wachspress stress ω^{W}. Since we fix the cone vertex at the origin, it disappeared from all our computations, which simplified the presentation.

3.1. **The Alexandrov matrix.** Let A be the *Alexandrov matrix* of P. We will use the following properties (see [4] or [6, Theorem 3.3]):

(A0) A is symmetric,
(A1) $A_{ij} = 0$ whenever $i \not\sim j$ and $i \neq j$,
(A2) $\ker A = \operatorname{span} \boldsymbol{n}$,
(A3) A has a unique positive eigenvalue.

The off-diagonal entries can be given explicitly:

$$A_{ij} = \frac{V_{ij}}{\sin(\theta_{ij})}. \tag{6}$$

Recall that $V_{ij} = \operatorname{vol}_{d-2}(F_i \cap F_j)$. We use $V_{ij} = 0$ whenever $i \not\sim j$.

Diagonal entries can be computed as follows: from (A2) we get $A\boldsymbol{n} = 0$, or equivalently, $\sum_j A_{ij} n_j = 0$ for every i. Taking the inner product with n_i yields

$$\sum_j A_{ij} \langle n_i, n_j \rangle = 0 \quad \Longrightarrow \quad A_{ii} = -\sum_{j:j\neq i} \frac{V_{ij} \langle n_i, n_j\rangle}{\sin(\theta_{ij})} = -\sum_{j:j\neq i} V_{ij} \cot(\theta_{ij}). \tag{7}$$

We also need the following two facts that follow from straightforward geometric computations (see also [12, Proposition 3.5]):

$$\sum_j h_j A_{ij} = (d-1) V_i, \tag{8}$$ $$\sum_{i,j} h_i h_j A_{ij} = d(d-1)\operatorname{vol}(P), \tag{9}$$

where $V_i := \operatorname{vol}_{d-1}(F_i)$.

3.2. **Proof scaffold.** The main work for proving Theorem 1 is spent on establishing the following two facts:[2]

$$\text{(I)} \quad \boldsymbol{h}^\top A (\dot n)_k = 0. \qquad \text{and} \qquad \text{(II)} \quad \sum_k (\dot n)_k^\top A (\dot n)_k = 0$$

where $\boldsymbol{h} = (h_1, ..., h_n)$ is the vector of facet heights. We can quickly finish the proof from here. First, recall

$$\boldsymbol{h}^\top A \boldsymbol{h} = \sum_{i,j} h_i h_j A_{ij} \overset{(9)}{=} d(d-1)\operatorname{vol}(P) > 0.$$

By (A3) the positive eigenvalue of A is unique, and hence A is negative semidefinite on the A-orthogonal complement $\boldsymbol{h}^{\perp A} := \{x \mid \boldsymbol{h}^\top A x = 0\}$. In particular, (I) gives $(\dot n)_k^\top A (\dot n)_k \leq 0$ for all k. With (II) this turns into $(\dot n)_k^\top A(\dot n)_k = 0$. Since A is semidefinite on $\boldsymbol{h}^{\perp A}$, this implies $(\dot n)_k \in \ker A$. From (A2) we get $(\dot n)_k \in \operatorname{span} \boldsymbol{n}$, or equivalently, $\dot n_i = M n_i$ for some matrix M. It remains to show that M is skew-symmetric.

The following argument is standard. Let $T = \frac{1}{2}(M^\top + M)$ be the symmetric part of A. Then

$$\begin{aligned} 2\langle T n_i, n_j \rangle &= \langle (M + M^\top) n_i, n_j \rangle \\ &= \langle M n_i, n_j \rangle + \langle n_i, M n_j \rangle = \langle \dot n_i, n_j \rangle + \langle n_i, \dot n_j \rangle \overset{(2)}{=} 0 \end{aligned}$$

whenever $j = i$ or $j \sim i$. So $T n_i$ is orthogonal to n_i and to all n_j with $j \sim i$. Since P is full-dimensional and convex, the latter contain a basis of $\mathbb{R}^d$. Hence $T n_i = 0$

[2] These two steps parallel the proof of prestress stability in [6, Section 4.1], though avoid any use of rigidity terminology.

for all i. But also the n_i contain a basis of $\mathbb{R}^d$. Hence $T = 0$, A has no symmetric part, and is therefore skew-symmetric.

3.3. **Proof of (I).** We show that $\mathbf{1}^\top A(\dot q)_k = 0$ for all k simultaneously:

$$\boldsymbol{h}^\top A\dot{\boldsymbol{n}} = \sum_{i,j} h_j A_{ij}\dot n_i = \sum_i \Big(\sum_j h_j A_{ij}\Big)\dot n_i \overset{(8)}{=} (d-1)\sum_i V_i \dot n_i$$

Consider the tensor field $X_i(x) := x \otimes \dot n_i$. Since $\langle n_i, \dot n_i\rangle = 0$, its divergence tangential to F_i is $\mathrm{div}_{F_i}(X_i) = \dot n_i$. Applying the divergence theorem in $(*)$ yields[3]

$$\begin{aligned} V_i\dot n_i = \int_{F_i} \mathrm{div}_{F_i}(X_i)\,\mathrm{d}x &\overset{(*)}{=} \int_{\partial F_i} X_i(x)\cdot \mathrm{d}n = \sum_{j:i\sim j}\int_{R_{ij}} x\langle \dot n_i, n_{ij}\rangle\,\mathrm{d}x \\ &= \sum_{j:i\sim j} I_{ij}\langle \dot n_i, n_{ij}\rangle, \quad \text{where } I_{ij} := \int_{R_{ij}} x\,\mathrm{d}x. \end{aligned}$$

Note that $I_{ij} = I_{ji}$. The proof finishes by summation over all facets:

$$\begin{aligned} \sum_i V_i\dot n_i = \sum_{\substack{i\neq j\\ i\sim j}} I_{ij}\langle \dot n_i, n_{ij}\rangle &= \sum_{\substack{i<j\\ i\sim j}} I_{ij}(\langle \dot n_i, n_{ij}\rangle + \langle \dot n_j, n_{ji}\rangle) \\ \text{subst. (4) and (5)} \longrightarrow \quad &= \sum_{\substack{i<j\\ i\sim j}} \frac{I_{ij}}{\sin(\theta_{ij})}\overbrace{(\langle \dot n_i, n_j\rangle + \langle n_i, \dot n_j\rangle)}^{=0 \text{ by } (2)} = 0. \end{aligned}$$

3.4. **Proof of (II).** The goal is to show $\sum_k (\dot q)_k^\top A(\dot q)_k = 0$. For each facet F_i, take the inner product of (3) with $\dot n_i$. Then sum over all facets. This yields

$$\sum_{i\neq j} V_{ij}\langle \dot n_{ij}, \dot n_i\rangle = -\sum_{i\neq j} \dot V_{ij}\langle n_{ij}, \dot n_i\rangle.$$

We will rewrite the left side to $\sum_k (\dot q)_k^\top A(\dot q)_k$ and show that the right side vanishes.

On the left side we compute

$$\begin{aligned} \sum_{i\neq j} V_{ij}\langle \dot n_{ij}, \dot n_i\rangle &\overset{(5)}{=} \sum_{i\neq j}\frac{V_{ij}}{\sin(\theta_{ij})}\langle \dot n_j - \langle n_i, n_j\rangle \dot n_i, \dot n_i\rangle \\ &= \sum_{i\neq j}\frac{V_{ij}}{\sin(\theta_{ij})}(\langle \dot n_j, \dot n_i\rangle - \langle n_i, n_j\rangle\|\dot n_i\|^2) \\ &= \sum_{i\neq j}\frac{V_{ij}}{\sin(\theta_{ij})}\langle \dot n_i, \dot n_j\rangle - \sum_i\sum_{j\sim i}\frac{V_{ij}\langle n_i, n_j\rangle}{\sin(\theta_{ij})}\|\dot n_i\|^2 \\ \text{subst. (6) and (7)} \longrightarrow \quad &= \sum_{i\neq j} A_{ij}\langle \dot n_i, \dot n_j\rangle + \sum_i A_{ii}\|\dot n_i\|^2 \\ &= \sum_{i,j} A_{ij}\langle \dot n_i, \dot n_j\rangle = \sum_k (\dot n)_k^\top A(\dot n)_k. \end{aligned}$$

[3] This argument is a highly compressed application of the vector-valued Schläfli formula from [8].

And on the right side we compute

$$\begin{aligned}
\sum_{i\neq j} \dot{V}_{ij}\langle n_{ij}, \dot{n}_i\rangle &\overset{(4)}{=} \sum_{i\neq j} \frac{\dot{V}_{ij}}{\sin(\theta_{ij})}\langle n_j - \langle n_i, n_j\rangle n_i, \dot{n}_i\rangle \\
&= \sum_{i\neq j} \frac{\dot{V}_{ij}}{\sin(\theta_{ij})}\big(\langle n_j, \dot{n}_i\rangle - \langle n_i, n_j\rangle \overbrace{\langle n_i, \dot{n}_i\rangle}^{=0 \text{ by } (1)}\big) \\
&= \sum_{i\neq j} \frac{\dot{V}_{ij}}{\sin(\theta_{ij})}\langle n_j, \dot{n}_i\rangle \\
&= \sum_{i<j} \frac{\dot{V}_{ij}}{\sin(\theta_{ij})}\overbrace{\big(\langle n_i, \dot{n}_j\rangle + \langle n_j, \dot{n}_i\rangle\big)}^{=0 \text{ by } (2)} = 0.
\end{aligned}$$

4. How the proof came about

In September 2026, correspondence with Arseniy Akopyan revealed that *Wachspress Geometry* (in particular, the study of the Izmestiev matrix) allows one to give a polyhedral proof of the (finite) Stoker conjecture [2]. This new proof builds on techniques developed by the author in [12]. Due to independent interest from the side of bar-joint rigidity, the central ideas of [12] have been developing towards an infinitesimal setting already since 2023. This included collaborations with Steven Gortler, Louis Theran and Robert Connelly [3], as well as Eleni Pachyli and Roman Prosanov [6]. It was natural to ask whether this line of work might permit a likewise elementary approach to the infinitesimal Stoker conjecture.

It quickly became clear that a proof could indeed be extracted from the recent result that coned polytope frameworks are prestress stable [6, Theorem 1.1]. The key step was the resolution of the weak stress-flex conjecture, which in turn was based on a vector-valued Schläfli formula first proved by Schlenker & Souam [8]. The known proof of the latter did however involve techniques that are not discrete-geometric [8]. Importantly then, one of the contributions of [6] was precisely to provide such a discrete-geometric proof for the formula. At last, a completely polyhedral proof of the infinitesimal Stoker conjecture appeared possible.

Subsequent investigations revealed that the derivation of the Schlenker-Souam-Schläfli formula could be significantly shortened, and for the purpose used in the proof here, even reduced to a simple application of the divergence theorem. This motivated the author to produce the condensed proof presented here.

Acknowledgments. This short proof is the result of numerous past collaborations, most notably, with Roman Prosanov, Eleni Pachyli, Steven Gortler, Louis Theran and Robert Connelly. I am also grateful for their input on earlier drafts of this article. I am thankful to Arseniy Akopyan, who brought to my attention that my past work contains a proof of the (finite) Stoker conjecture, which triggered this development, and to Karim Adiprasito, who provided exposition through his blog post.

Funding. The author is funded by the SPP 2458 "Combinatorial Synergies" (project ID 539851419), funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation).

Statement on AI use. Generative AI was used during the development and preparation of this article. It assisted in checking calculations, identifying gaps and possible simplifications in preliminary versions of the argument. Some elementary reformulations and shortcuts in the final proof arose from these interactions. The overall proof strategy and mathematical development are the author's. The entire article was written by the author, and all arguments and references appearing in the article were independently checked by the author, who takes full responsibility for the mathematical content.

References

[1] K. A. Adiprasito. Geometry and the simplex: Results, questions and ideas. *European Mathematical Society Magazine*, (118):28–33, 2020.

[2] K. A. Adiprasito. Stoker, again, Sept. 2026. Unsolicited Nonsense.

[3] R. Connelly, S. J. Gortler, L. Theran, and M. Winter. The stress-flex conjecture. *arXiv preprint arXiv:2404.15590*, 2024.

[4] I. Izmestiev. The Colin de Verdière number and graphs of polytopes. *Israel Journal of Mathematics*, 178(1):427–444, 2010.

[5] R. Mazzeo and G. Montcouquiol. Infinitesimal rigidity of cone-manifolds and the Stoker problem for hyperbolic and Euclidean polyhedra. *Journal of Differential Geometry*, 87(3):525–576, 2011.

[6] E. Pachyli, R. Prosanov, and M. Winter. Second-order rigidity of coned polytope frameworks and the stress-flex conjecture from a vector-valued Schläfli formula. *arXiv preprint arXiv:2607.14878*, 2026.

[7] J.-M. Schlenker. Dihedral angles of convex polyhedra. *Discrete & Computational Geometry*, 23(3):409–417, 2000.

[8] J.-M. Schlenker and R. Souam. Higher Schläfli formulas and applications. II. Vector-valued differential relations. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 068, 44, 2008.

[9] J. J. Stoker. Geometrical problems concerning polyhedra in the large. *Communications on pure and applied mathematics*, 21(2):119–168, 1968.

[10] H. Weiss. Local rigidity of 3-dimensional cone-manifolds. *Journal of Differential Geometry*, 71(3):437–506, 2005.

[11] H. Weiss. The deformation theory of hyperbolic cone–3–manifolds with cone-angles less than 2π. *Geometry & Topology*, 17(1):329–367, 2013.

[12] M. Winter. Rigidity, tensegrity, and reconstruction of polytopes under metric constraints. *International Mathematics Research Notices*, 2024(9):7721–7747, 2024.