STOKER'S THEOREM FROM POLYTOPAL TENSEGRITIES

MARTIN WINTER

ABSTRACT. This document gives a short derivation of Stoker's theorem starting from the results of [5]. It is distilled from a document produced by Arseniy Akopyan together with an LLM.

The following is an inequality version of Stoker's conjecture (which has been a theorem since 2023 [1,4]):

Theorem 1. *Let $P, P' \subset \mathbb{R}^d$ be two combinatorially equivalent polytopes. Let $n_i, n_i' \in \mathbb{R}^d$ be the outwards pointing unit normal vectors of the i-th facet in P and P' respectively. Suppose that whenever two facets i and j are adjacent, then their dihedral angles satisfy*

$$\pi - \arccos\langle n_i, n_j\rangle \;=:\; \theta_{ij} \;\le\; \theta'_{ij} \;:=\; \pi - \arccos\langle n'_i, n'_j\rangle.$$

Then $n_i = Tn_i'$ for all facets i and some fixed orthogonal transformation T.

THE PROOF

We may assume $0 \in \operatorname{int}(P)$ and $0 \in \operatorname{int}(P')$. Let $h_i, h_i' > 0$ the heights of the i-th facet over the origin in P and P' respectively. The vertices of the polar dual P° are given by $p_i = n_i/h_i$. Let $G = (V, E)$ be the graph of P°.

Now consider the points $p_i' := n_i'/h_i$. Notice that the denominator is h_i, not h_i', and the p_i' are therefore not the vertices of the polar dual of P', and in fact, not necessarily the vertices of any realization of P°.

With this we have $\|p_i\| = \|p_i'\|$, and from the increasing dihedral angles also $\|p_i - p_j\| \ge \|p_i' - p_j'\|$ for all edges $ij \in E$. We would like to invoke the following specialized version of [5, Lemma 4.3]:

Lemma 2. *Let P° have vertices p_i and graph $G = (V, E)$. If $p' \colon V \to \mathbb{R}^d$ satisfies*

(i) $\|p_i\| = \|p_i'\|$ *for all* $i \in V$,
(ii) $\|p_i - p_j\| \ge \|p_i' - p_j'\|$ *for all* $ij \in E$,
(iii) $0 \in \phi(\operatorname{int}(P^\circ))$, *where ϕ is the Wachspress map $\phi : P^\circ \to \operatorname{conv}\{p_i' \mid i \in V\}$,*

then $p_i = Tp_i'$ for some orthogonal transformation T.

The missing condition *(iii)* is technical and we deal with it momentarily. Its interpretation is that the origin is "properly surrounded" by the points p_i'. If we assume it, the lemma clearly gives $n_i = Tn_i'$ for all facets and proves Theorem 1.

We do not give a full definition of the Wachspress map but merely recall that it is a continuous map $\phi\colon P^\circ \to \operatorname{conv}\{p_i' \mid i \in V\}$ with the following property:

($*$) if $f \subseteq P^\circ$ is a face then $\phi(f) \subseteq \operatorname{conv}\{p_i' \mid p_i \in f\}$.

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Let's first look at the case $h_i = h_i'$: then $p_i' = n_i'/h_i'$ would be the vertices of the polar dual P'°, and hence the vertices of a realization of P°. Then (iii) follows from a simple degree argument given in [5, Section 4.4]: from $(*)$ follows that ϕ maps each face of P° to its corresponding face in P'°, and hence restricts to a degree-1 map $\partial P^\circ \to \partial P'^\circ$ between the boundaries. Since $0 \in \mathrm{int}(P'^\circ)$, Brouwer's fixed point theorem yields $0 \in \phi(\mathrm{int}(P^\circ))$.

It remains to argue that this stays true for $h_i \neq h_i'$. For this, consider the linear homotopy

$$p_i'(t) := \Big(\frac{t}{h_i} + \frac{1-t}{h_i'}\Big) n_i', \quad t \in [0,1]$$

moving p_i' along the ray spanned by n_i'. As argued above, the statement is true for $t = 0$. It therefore suffices to show that $\phi(\partial P^\circ)$ never crosses the origin as $t \to 1$ because then the degree of $\phi(\partial P^\circ)$ around the origin stays nonzero, and hence, $0 \in \phi(\mathrm{int}(P^\circ))$ also at $t = 1$.

For sake of contradiction, assume that $\phi(\partial P^\circ)$ crosses the origin at time t, that is, there is a proper face $f \subset P^\circ$ with

$$0 \in \phi(f) \overset{(*)}{\subseteq} \mathrm{conv}\{p_i'(t) \mid p_i \in f\}.$$

Let H be a supporting hyperplane for f in P'°. Moving H to the origin yield a hyperplane that has $\{p_i'(0) \mid p_i \in f\}$ strictly on one side. This stays true throughout the homotopy since all scaling factors are strictly positive. But then 0 cannot have been in the convex hull of these $p_i'(t)$. $\square$

Clearly major work of the proof happens hidden inside of [5, Lemma 4.3] (and actually, [5, Theorem 2.3]). A main ingredient is a result by Ivan Izmestiev [3] (or [5, Theorem 3.3]) on the Colin de Verdiére matrices of polytope graphs, which in turn builds on the second Minkowski inequality and Bol's condition for equality. With Ivan's result in place, short proofs are possible, see [5, Appendix F] and another improved presentation found together with Robert Connelly, Steven Gortler and Louis Theran [2].

References

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Max-Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Gernany

Email address: martin.winter@mis.mpg.de