SECOND-ORDER RIGIDITY OF CONED POLYTOPE FRAMEWORKS AND THE STRESS-FLEX CONJECTURE

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Abstract. A *coned polytope framework* (CPF) is the bar-joint framework obtained from the 1-skeleton of a convex polytope by coning over some interior point. It was recently shown that CPFs are rigid, though the exact order of rigidity remained open. In this paper we introduce the Wachspress stress and use it to show that the CPFs are prestress stable, in particular, second-order rigid. To this end, we resolve the stress-flex conjecture in the case of the Wachspress stress by identifying its dual formulation as a corollary of a vector-valued Schläfli-type formula introduced by Schlenker and Souam. We give a new, purely discrete-geometric proof of this generalized Schläfli formula.

1. Introduction

1.1. **Rigidity of coned polytope frameworks.** A *coned polytope framework* (or *CPF* for short) is a bar-joint framework that is constructed from the 1-skeleton of a convex polytope $P \subset \mathbb{R}^d$ by inserting additional bars (the *cone bars*) between the vertices of P and some interior point $p_\star \in \text{int}(P)$. It was recently proven that the CPFs are rigid [20, Theorem 4.5]. That is, any sufficiently small perturbation of the 1-skeleton of P preserving the edge lengths and the distances to $p_\star$ is congruent to P. This result is surprising since polytope skeleta can be rather sparse. It is also somewhat unsatisfying as the proof gives no indication of the actual order of rigidity. In particular, many CPFs are not first-order rigid (*e.g.* a first-order flex is shown in Figure 1), and the question of second-order rigidity remained open [20, Question 5.2.]. We resolve this question:

Theorem 1.1. *Coned polytope frameworks are prestress stable (and hence, second-order rigid).*

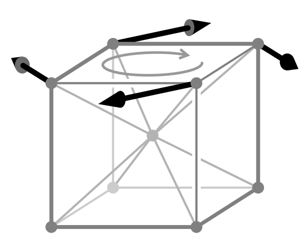

Figure 1. A coned polytope framework of the standard cube together with a first-order flex that (infinitesimally) twists the top square.

Prestress stability is a stronger and more convenient form of second-order rigidity [8]. It requires to pick a distinguished stress for the framework. We shall show that

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the underlying polytopal structure of a CPF gives rise to such a special stress – the *Wachspress stress* $\boldsymbol{\omega}^{\mathrm{W}}$ – of which we give a detailed account.

For the proof of prestress stability we follow the path laid out in an earlier note: in [7] it was shown that prestress stability would be a consequence of a curious property of the Wachspress stress called *stress-flex orthogonality* (*cf.* equation (1.1) below). The question for whether $\boldsymbol{\omega}^{\mathrm{W}}$ has this property became known as the *weak stress-flex conjecture*. We resolve this conjecture affirmatively:

Theorem 1.2. *If $\dot{\boldsymbol{p}}$ is a first-order motion of a CPF with $\dot{p}_\star = 0$, then*

$$\sum_{i\neq\star} \omega^{\mathrm{W}}_{\star i}\dot{p}_i = 0. \tag{1.1}$$

We observe that the special structure of the Wachspress stress makes (1.1) dual to a generalization of the classical Schläfli formula first proven by Schlenker and Souam [18]. We give an elementary and self-contained proof of the relevant fragment of this generalization.

Theorem 1.2 resolves a special case of the *(strong) stress-flex conjecture* – an analogous conjecture about general stresses and piecewise linear surfaces. We also briefly discuss approaches to this conjecture.

1.2. **Generalized Schläfli formulas.** Let $P^t \subset \mathbb{R}^d$, $t \in [0,1]$, be a differentiable 1-parameter family of convex polytopes having the same facets. In other words, the outward-oriented hyperplanes containing the facets of $P := P^0$ move differentiably (as elements of the co-Euclidean d-space, which is the space of oriented hyperplanes of $\mathbb{R}^d$) and define polytopes P^t, which may have different combinatorics. For a face σ of P, we denote its volume by V_σ. When we want to specify the dimension of a face of P, we write σ_k for a face of dimension k. For faces $\sigma_k \subset \sigma_{k+2}$, we denote by $\theta^{\sigma_{k+2}}_{\sigma_k}$ the exterior dihedral angle of σ_k in σ_{k+2}. The Euclidean version of the celebrated Schläfli formula states that

Theorem 1.3 ([1, Chapter 7.2.2], [13])**.**

$$\sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} = 0\ .$$

Here the sum is over all $(d-2)$-faces σ_{d-2} of P and the derivative is taken at $t=0$.

The Schläfli formula extends to polytopes in the spherical d-space $\mathbb{S}^d$ and in the hyperbolic d-space $\mathbb{H}^d$, where in the right-hand side zero is replaced by $\pm\dot{V}_{\sigma_d}$ respectively. Here V_{σ_d} is the volume of P. The Schläfli formula is known for its broad spectrum of applications, ranging from the study of isometric embeddings and their rigidity [4,14], to discrete conformality [5,11,15] and to geometrization results [6,12], to name a few. It was generalized in a plethora of directions. The relevant one for our paper is its vector-valued extension due to Schlenker and Souam [18]. For a face σ of P, we denote by b_σ its barycenter. For faces $\sigma_k \subset \sigma_{k+1}$, we denote by $n^{\sigma_{k+1}}_{\sigma_k}$ the unit normal vector from σ_{k+1} towards σ_k. The mentioned result is

Theorem 1.4 ([18, Theorem 1.11])**.**

$$\sum_{\sigma_{d-1}} \dot{V}_{\sigma_{d-1}} n^{\sigma_d}_{\sigma_{d-1}} = \sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} b_{\sigma_{d-2}}\ .$$

The left-hand side in Theorem 1.4 is clearly translation-invariant. It follows that for an arbitrary non-zero vector $a \in \mathbb{R}^d$, we have

$$\left(\sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} \right) a = 0 \ ,$$

from which it follows that $\sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} = 0$. Thereby, Theorem 1.4 implies Theorem 1.3. Theorem 1.4 relates the Schläfli formula to another well-known result in polyhedral geometry, the Minkowski theorem:

Theorem 1.5 ([19, Section 8.2.1])**.**

$$\sum_{\sigma_{d-1}} V_{\sigma_{d-1}} n^{\sigma_d}_{\sigma_{d-1}} = 0 \ .$$

It turns out that the dual statement to Theorem 1.2 from the previous section is the following corollary of Theorem 1.4:

Corollary 1.6. *If for all σ_{d-2} we have $\dot{\theta}^{\sigma_d}_{\sigma_{d-2}} = 0$, then*

$$\sum_{\sigma_{d-1}} \dot{V}_{\sigma_{d-1}} n^{\sigma_d}_{\sigma_{d-1}} = \sum_{\sigma_{d-1}} V_{\sigma_{d-1}} \dot{n}^{\sigma_d}_{\sigma_{d-1}} = 0 \ .$$

Similarly to the classical Schläfli formula, in [18] Schlenker and Souam also supply the spherical and hyperbolic versions of Theorem 1.4. In terms of the proof strategy, actually they begin from the spherical and hyperbolic ones, and then deduce from them the Euclidean case by means of a blow-up argument. The spherical and hyperbolic cases are obtained in [18] by approximating the polytopes with smooth convex bodies, establishing for them smooth analogues of vector-valued Schläfli-type formulas and applying a limiting argument. Thereby, to establish Theorem 1.4 in [18], a limiting construction is used twice.

The second goal of our article is to give a different, more direct and discrete-geometric proof of Theorem 1.4, not relying on approximation arguments. Our proof is based on the orthoscheme decomposition, also introduced by Schläfli. We will employ the mentioned classical theorems of Schläfli and Minkowski, as well as another result of Schlenker and Souam from [18], for which we will also provide a new proof.

Theorem 1.7 ([18, Theorem 1.1])**.** *Let $n_1, \dots, n_k$ be an ordered k-tuple of points on $\mathbb{S}^2 \subset \mathbb{R}^3$ with $n_i \neq \pm n_{i-1}$, α_i be the oriented angle $n_{i-1}n_in_{i+1}$ (the angle between spherical segments) with $\alpha_i \neq 0, \pm\pi$, $i = 1, \dots, k$. Let m_i be the dual point to the oriented segment n_in_{i+1} and β_i be the oriented angle $m_{i-1}m_im_{i+1}$ (hence, β_i is equal to the oriented length of n_in_{i+1}). Then for an infinitesimal deformation of n_i we get*

$$\sum_i \dot{\alpha}_i n_i = \sum_i \dot{\beta}_i m_i \ .$$

The case of Theorem 1.7 corresponding to the isometric deformations, *i.e.,* when all $\dot{\beta}_i = 0$, is quite frequent in the polyhedral literature, see, *e.g.* [2, Section 10.1], [9, Section 3.8], [16, Theorem A$_\mathrm{S}$]. As we already mentioned, the full case is proven in [18] by Schlenker and Souam. However, it does not have a separate proof there, the authors rather deduce it from their very general results. In particular, their proof requires an approximation of polyhedral curves by smooth curves and a limiting argument. In Appendix A we will give a streamlined, self-contained proof of Theorem 1.7, generalizing the approach of Glück from [9]. A different proof of Theorem 1.7 implicitly follows from [10], although the statement is not explicitly formulated there and [10] uses a very different language from ours. It is interesting that in [10] Theorem 1.7 is used in the context of classification of constant mean curvature surfaces in Euclidean 3-space.

In [18] Schlenker and Souam also prove higher-signature vector-valued generalizations of Theorem 1.4, which extend their earlier higher-signature scalar analogues of the Schläfli formula obtained in [17]. It is of interest, whether our methods can be also applied to give new proofs of these results, as well as of their spherical and hyperbolic counterparts.

Structure of the paper. In Section 2 we recall the basic notions of rigidity theory. This includes the relevant tools from the first- and second-order theory of framework rigidity, in particular, the notion of *prestress stability*.

In Section 3 we define the *Wachspress stress* that exists for every coned polytope framework and will play the central role in establishing prestress stability. We explain its relation to the Wachspress coordinates and the Izmestiev matrix, and we derive its relevant properties.

In Section 4 we first recall how stress-flex orthogonality of the Wachspress stress implies that CPFs are prestress stable (Section 4.1). Subsequently we derive Theorem 1.2 from Theorem 1.4 (Section 4.2). This finalizes the proof of Theorem 1.1.

In Section 5 we provide our new proof of Theorem 1.4.

In **??** we discuss approaches to the (strong) stress-flex conjecture.

In Appendix A we furnish a proof of Theorem 1.7.

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2. Polytopes, frameworks and rigidity

Throughout the text let $P \subset \mathbb{R}^d$ denote a (convex) polytope spanned by its finitely many vertices $p_1, ..., p_n$. Let $p_\star \in \mathrm{int}(P)$ be some interior point of P.

By $G_P = (V_P, E_P)$ we denote the *vertex-edge graph* of P, that is, $V_P = \{1, ..., n\}$, with $ij \in E_P$ if and only of p_i and p_j form an edge in P. Then

$$G_P^\star = (V_P^\star, E_P^\star) = (V_P \cup \{\star\}, E_P \cup E_c)$$

denotes the *coned vertex-edge graph*, that is, the graph obtained from G_P by adding a new *cone vertex* $\star \in V$ adjacent to all vertices in V_P along new *cone edges* E_c. For distinction, the edges in E_P we shall call *polytope edges*.

A d-dimensional *bar-joint framework* (or *framework* for short) is a pair $(G, \boldsymbol{p})$ consisting of a graph $G = (V, E)$ and an embedding map $\boldsymbol{p}\colon V \to \mathbb{R}^d$. One thinks of the edges of G as straight line segments between the embedded vertices. For example, we can treat the polytope's *1-skeleton* as a framework of the vertex-edge graph G_P. It is common to use the terms *joint* and *bar* for the embeddings of vertices and edges to distinguish from their combinatorial meaning within the graph G.

Given P and $p_\star$ as above, the corresponding *coned polytope framework* (or *CPF* for short) is the framework $(G_P^\star, \boldsymbol{p})$ of the coned vertex-edge graph that embeds the polytope's vertices and the cone point in the obvious way.

For the rest of this section we recall the necessary aspects of rigidity theory with some comments on how this applies to the theory of CPFs.

2.1. **Motions and flexes.** Rigidity theory studies deformations of frameworks that preserve bar lengths. Formally, a *motion* of $\boldsymbol{p}$ is a continuous 1-parameter family $\boldsymbol{p}^t$, $t \in [0,1]$ of embeddings with $\boldsymbol{p}^0 = \boldsymbol{p}$ and $\|p_i^t - p_j^t\|$ constant throughout for all edges $ij \in E$. A motion is *trivial* if it preserves *all* pairwise distances between vertices, not only along edges. Trivial motions are induced by continuous families of global isometries. A non-trivial motion is called a *flex*. A framework is *flexible* if it has a flex, and is said to be *rigid* otherwise.

Using the tools of Wachspress Geometry it was proven in [20, Theorem 4.5] that CPFs are rigid. This is surprising since polytope skeleta can be quite sparse (see Remark 2.1 below). Rigidity theory has tools to quantify this discrepancy by investigating the order at which rigidity sets in. For CPFs this leads to the open questions we study in this article. In the next two section we recall the first- and second-order theory of rigidity.

2.2. **First-order theory.** First-order theory studies deformations of a framework that preserves bar lengths in the first order, that is, for $ij \in E$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|p_i^t - p_j^t\|\big|_{t=0} = 0 \quad\Longrightarrow\quad 0 = \tfrac{\mathrm{d}}{\mathrm{d}t}\|p_i - p_j\|^2 = \big\langle p_i - p_j, \tfrac{\mathrm{d}}{\mathrm{d}t}p_i - \tfrac{\mathrm{d}}{\mathrm{d}t}p_j\big\rangle. \tag{2.1}$$

For convenience we drop the superscript t if we evaluate at $t = 0$.

This motivates the following definition: a map $\dot{\boldsymbol{p}} : V \to \mathbb{R}^d$ is a *first-order motion* if it satisfies

$$\langle p_j - p_i, \dot p_j - \dot p_i\rangle = 0, \quad \text{for all } ij \in E. \tag{2.2}$$

Every differentiable family as in (2.1), in particular every motion, gives rise to such a first-order motion via $\dot{\boldsymbol{p}} := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}^t|_{t=0}$. The converse is true as well: every first-order motion comes from differentiating a family as in (2.1), though not necessarily from differentiating a motion.

It is possible to characterize the first-order motions that are obtained from differentiating trivial motions. They are precisely of the form

$$\dot{\boldsymbol{p}} = S\boldsymbol{p} + t, \tag{2.3}$$

where $S \in \mathbb{R}^{d\times d}$ is *skew-symmetric* and $t \in \mathbb{R}^d$. Such first-order motions are called *trivial* and they exist for every framework irrespective of structural details. If non-trivial, they are called *first-order flexes*. A framework is *first-order flexible* if it has first-order flexes, and *first-order rigid* otherwise.

A major benefit of first-order theory is that it reduces rigidity analysis to linear algebra and thereby provides practical necessary criteria for the existence of a flex: if a framework is first-order rigid, then it is rigid [3]. It moreover allows us to make heuristic statements about the expected rigidity behavior of a system. For example, the following argument suggests that many CPFs should not be rigid:

Remark 2.1. Let $P \subset \mathbb{R}^d$ be a *simple* polytope, that is, each vertex has degree d. We perform the naive count "degrees of freedom (DOFs) minus constraints" on its coned framework. First, ignoring the bar constraints, each joint has d independent ways to move. This yields $d(|V|+1)$ DOFs. Heuristically, we expect to lose exactly one DOF from each bar constraint. There are $|E|$ bars from the polytope edges, and $|V|$ bars from the cone edges. For a d-regular graph we have $2|E| = d|V|$. The total count therefore gives

$$\#\text{DOFs} - \#\text{constraints} = d(|V|+1) - (|V|+|E|) = |V|(\tfrac{d}{2}-1) + d.$$

In dimension d there are d translations and $\binom{d}{2}$ rotations, contributing $d + \binom{d}{2}$ unavoidable freedoms. A straightforward computation shows that for $d \geq 3$ and $|V| \geq \frac{d(d-1)}{d-2} \geq 6$ the above count exceeds the number of trivial motions and therefore suggests that the CPF is first-order flexible.

For example, if P is the 3-dimensional cube (which has $|V| = 8$), we expect to find a single first-order flex. An exact computation confirms this (see also Figure 1).

2.3. **Second-order theory.** Second-order theory studies continuous deformations that preserve bar length up to second order. It is employed for the analysis of frameworks that fail the first-order rigidity tests, and hence a natural choice for the study of CPFs. Here we use an alternative (but equivalent) formulation of the theory developed by Connelly and Whiteley [8] that expresses second-order rigidity using the bilinear interaction between first-order flexes and stresses.

An *equilibrium stress* (or *stress* for short) of a framework $(G, \boldsymbol{p})$ is a map $\boldsymbol{\omega}\colon E \to \mathbb{R}$ that satisfies

$$\sum_{j:ij\in E(G)} \omega_{ij}(p_j - p_i) = 0, \quad \text{for all } i \in V.$$

The framework is *second-order rigid* if for each first-order flex $\dot{\boldsymbol{p}}$ there is a stress $\boldsymbol{\omega}$ that *blocks* $\dot{\boldsymbol{p}}$, which means

$$Q_{\boldsymbol{\omega}}(\dot{\boldsymbol{p}}) := \sum_{ij\in E} \omega_{ij}\|\dot{p}_i - \dot{p}_j\|^2 > 0. \tag{2.4}$$

One can show that $Q_{\boldsymbol{\omega}}(\dot{\boldsymbol{p}}) = 0$ for every trivial first-order motion $\dot{\boldsymbol{p}}$, *i.e.,* no trivial motion can be blocked by a stress.

Establishing second-order rigidity can be difficult since the blocking stress might be different for each first-order flex. The following is a stronger but more convenient notion: a framework is *prestress stable* if there is a *single* stress $\boldsymbol{\omega}$ that blocks *all* first-order flexes. The notion and name originate in structural engineering: if a physical framework is constructed with a stress built into the structure, then any deformation in the direction of a first-order flex increases the energy expressed by (2.4), *i.e.,* is energetically unfavorable. It holds

$$\text{first-order rigid} \;\Rightarrow\; \text{prestress stable} \;\Rightarrow\; \text{second-order rigid} \;\Rightarrow\; \text{rigid}.$$

None of the inverse implications holds in general [8].

Lastly, we include an alternative way to express (2.4): recall that the *stress matrix* $\Omega \in \mathbb{R}^{V\times V}$ of a stress $\boldsymbol{\omega}$ has entries

$$\Omega_{ij} := \begin{cases} \omega_{ij} & \text{if } ij \in E \\ -\sum_{k:ik\in E} \omega_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

In particular, Ω is symmetric and has row and column sum zero. If we interpret $\dot{\boldsymbol{p}}$ as a $(d \times V)$-matrix, one can check that

$$\operatorname{tr}(\dot{\boldsymbol{p}}\Omega\dot{\boldsymbol{p}}^\top) = \sum_{ij\in E} \omega_{ij}\|\dot{p}_i - \dot{p}_j\|^2 = Q_{\boldsymbol{\omega}}(\dot{\boldsymbol{p}}). \tag{2.5}$$

3. The Wachspress stress

To certify prestress stability of a coned polytope framework we need to choose a suitable stress. Stresses are an expression of redundancies in frameworks; and since a CPF can be quite sparse (*cf.* Remark 2.1), one might not expect to find a stress at all. Against these odds, in this section we demonstrate how the piecewise linear boundary structure of a polytope gives rise to a distinguished stress for each CPF – the *Wachspress stress.*

As before, let $P \subset \mathbb{R}^d$ be a convex polytope with interior point $p_\star \in \operatorname{int}(P)$. Let $(G, \boldsymbol{p})$ the corresponding coned polytope framework. By

$$(P - p_\star)^\circ := \{y \in \mathbb{R}^d \mid \langle y, x - p_\star\rangle \le 1 \text{ for all } x \in P\}$$

we denote the polar dual of P, obtained by polarizing at $p_\star$. Let $F_i^\diamond$ be the facet of the polar dual that is dual to the vertex p_i of P. Likewise, let $F_{ij}^\diamond$ be the ridge (*i.e.,* codimension-2 face) of the polar dual that is dual to the edge ij of P. Finally, let $C_i := F_i^\diamond \vee p_\star$ and $C_{ij} := F_{ij}^\diamond \vee p_\star$ be the cones over $F_i^\diamond$ and $F_{ij}^\diamond$ respectively, where $\vee$ expresses the convex hull with the cone point.

The *Wachspress stress* $\boldsymbol{\omega}^{\mathrm{w}}$ (also *Izmestiev stress*) of $(G, \boldsymbol{p})$ is given by

$$\omega^{\mathrm{w}}_{\star i} := \frac{\operatorname{vol}(F_i^\diamond)}{\|p_i - p_\star\|} \ \ (\text{for } i \ne \star) \qquad \omega^{\mathrm{w}}_{ij} := -\frac{\operatorname{vol}(C_{ij})}{\|p_j - p_i\|} \ \ (\text{for } i, j \ne \star) \tag{3.1}$$

That this is indeed a stress of the framework follows from a repeated application of the *Minkowski balancing property*:

Theorem 3.1 (Minkowski balancing property, see *e.g.* [19, Section 8.2.1])**.** *Let $P \subset \mathbb{R}^d$ be a polytope. For a facet F of P, let $n_F \in \mathbb{R}^d$ be the (outwards pointing) unit normal vector. Then*

$$\sum_F n_F \operatorname{vol}(F) = 0, \tag{3.2}$$

where the sum is over all facets F of P.

Lemma 3.2. *The Wachspress stress $\boldsymbol{\omega}^{\mathrm{w}}$ is a stress of the coned polytope framework.*

Proof. The stress equilibrium at the cone vertex $\star$ is obtained by applying Minkowski's balancing property to $(P - p_\star)^\circ$. Note that the unit normal vector n_i of facet $F_i^\diamond \subset (P - p_\star)^\circ$ points in the direction $p_i - p_\star$:

$$\sum_{i\ne\star} \omega_{\star i}(p_i - p_\star) = \sum_{i\ne\star} \frac{\operatorname{vol}(F_i^\diamond)}{\|p_i - p_\star\|}(p_i - p_\star) = \sum_{i\ne\star} n_i \operatorname{vol}(F_i^\diamond) = 0.$$

The stress equilibrium at a non-cone vertex $i \ne \star$ is obtained by applying Minkowski's balancing property to the cone $C_i \subset (P - p_\star)^\circ$. The facets of C_i are $F_i^\diamond$ and the cones C_{ij} for $ij \in E_P$ (see Figure 2). Note that the normal vector n_{ij} at the facet C_{ij} points in the direction of $p_j - p_i$. With this, we obtain

$$\begin{aligned}\sum_{j:ij\in E}\omega_{ij}(p_j-p_i) &= \omega_{\star i}(p_\star - p_i) + \sum_{j:ij\in E_P}\omega_{ij}(p_j-p_i)\\ &= -\frac{\mathrm{vol}(F_i^\diamond)}{\|p_i-p_\star\|}(p_i-p_\star) - \sum_{j:ij\in E_P}\frac{\mathrm{vol}(C_{ij})}{\|p_j-p_i\|}(p_j-p_i)\\ &= -\Big(n_i\,\mathrm{vol}(F_i^\diamond) + \sum_{j:ij\in E_P} n_{ij}\,\mathrm{vol}(C_{ij})\Big) = 0. \qquad \square\end{aligned}$$

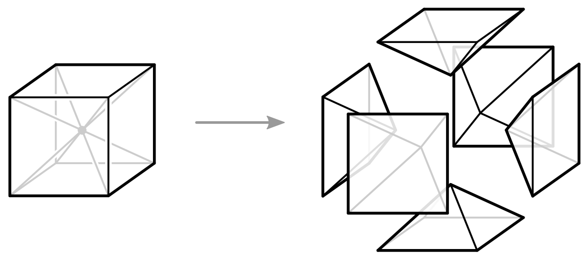

Figure 2. Decomposition of a pointed cube into its facial pyramids. The entries of the Wachspress stress are derived from the face volumes in this decomposition.

The name *Wachspress stress* derives from the fact that the entries $\omega^{\mathrm{w}}_{\star i}$ on the cone edges are (up to a normalization factor) precisely the *Wachspress coordinates* $\alpha \in \mathbb{R}^n$ of the point $p_\star$ in P. Likewise, the entries ω^{w}_{ij} on the polytope edges correspond to the entries of the *Izemstiev matrix* $M \in \mathbb{R}^{n\times n}$ for the point $p_\star$ in P. A brief yet sufficient introduction to both the Wachspress coordinates and the Izemstiev matrix can be found in [20, Section 3.1 – 3.3]. We recall the essential facts.

Theorem 3.3 ([20, Theorem 3.3])**.** *Given a polytope $P \subset \mathbb{R}^d$ with interior point $p_\star \in \mathrm{int}(P)$, there exists a symmetric matrix $M \in \mathbb{R}^{n\times n}$ (the Izmestiev matrix of $p_\star$ in P) with the following properties:*

(*i*) $M_{ij} > 0$ *whenever* $ij \in E_P$,
(*ii*) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E_P$,
(*iii*) M *has a unique positive eigenvalue (i.e., of multiplicity one), and*
(*iv*) $\ker M = \mathrm{span}(p_1 - p_\star, ..., p_n - p_\star)^\top$.

Explicit expressions are known for the entries of the Izmestiev matrix. The off-diagonal "on-edge" entries are typically given in the form [20, Equation (3.4)]:

$$M_{ij} = \frac{\mathrm{vol}(F_{ij}^\diamond)}{\|p_i-p_\star\|\|p_j-p_\star\|\sin\measuredangle(p_i-p_\star, p_j-p_\star)}, \quad \text{if } ij \in E_P. \tag{3.3}$$

The diagonal entries can be computed from the following relation to the (unnormalized) *Wachspress coordinates* $\tilde\alpha_i := \mathrm{vol}(F_i^\diamond)/\|p_i - p_\star\| = \omega_{\star i}$:

Theorem 3.4 ([20, Corollary 3.6])**.** $\sum_j M_{ij} = \tilde\alpha_i$ *for all* $i \in \{1, ..., n\}$.

We can now express the *stress matrix* Ω^{w} of ω^{w} in terms of these quantities.

Lemma 3.5. Ω^{w} *has the block form*

$$\Omega^{\mathrm{w}} := \begin{array}{c} \begin{array}{ccc} 1 & \cdots\ \ n & \star \end{array} \\ \left[\begin{array}{c|c} -M & \tilde{\alpha} \\ \hline \tilde{\alpha}^\top & -v \end{array}\right] \end{array} \begin{array}{c} 1 \\ \vdots \\ n \\ \star \end{array},$$

where M is the Izemstiev matrix of $p_\star$ in P, $\tilde{\alpha}$ are the (unnormalized) Wachspress coordinates of $p_\star$ in P, and $v := \mathrm{vol}(P - p_\star)^\circ$.

Proof. The identity holds for the entries in the $\tilde{\alpha}$-blocks by definition.

We next check the M-block. We first verify that M_{ij} from (3.3) agrees with $-\omega^{\mathrm{w}}_{ij}$ from (3.1) whenever $ij \in E$. This amounts to checking

$$-\omega^{\mathrm{w}}_{ij} = \frac{\mathrm{vol}(C_{ij})}{\|p_j - p_i\|} \stackrel{?}{=} \frac{\mathrm{vol}(F^\diamond_{ij})}{\|p_i - p_\star\|\|p_j - p_\star\| \sin\measuredangle(p_i - p_\star, p_j - p_\star)} = M_{ij}. \tag{3.4}$$

The argument is elementary geometric: let h_{ij} be the altitude of the triangle $\triangle p_\star p_i p_j$ at the vertex $p_\star$, *i.e.*, h_{ij} is the distance of $p_\star$ from the affine span of the edge $e_{ij} := \mathrm{conv}\{p_i, p_j\} \subset P$. Since $F^\diamond_{ij} \subset P^\circ$ is dual to the edge e_{ij}, the distance of its affine span from $p_\star$ is h_{ij}^{-1}. In particular, $\mathrm{vol}(C_{ij}) = \mathrm{vol}(F^\diamond_{ij}) h_{ij}^{-1}$. After substituting this into the target equation (3.4) and comparing both sides, it remains to verify

$$h_{ij} = \frac{\|p_i - p_\star\|\|p_j - p_\star\| \sin\measuredangle(p_i - p_\star, p_j - p_\star)}{\|p_j - p_i\|}.$$

But the right side is a well-known expression for the altitude in a triangle.

The identity on the diagonal entries of the M-block now follow from Theorem 3.4 and the fact that stress matrices have zero row sum.

It remains to verify the bottom-right entry. Note that the distance of (the affine span of) the facet $F^\diamond_i$ from $p_\star$ is $\|p_i - p_\star\|^{-1}$. Hence $\mathrm{vol}(C_i) = \mathrm{vol}(F^\diamond_i)\|p_i - p_\star\|^{-1} = \tilde{\alpha}_i$. Since Ω^{w} has zero row sums, the bottom-right entry computes to

$$v = \sum_i \tilde{\alpha}_i = \sum_i \frac{\mathrm{vol}(F^\diamond_i)}{\|p_i - p_\star\|} = \sum_i \mathrm{vol}(C_i) = \mathrm{vol}(P - p_\star)^\circ,$$

where for the last identity we used that the cones C_i have disjoint interior and cover $(P - p_\star)^\circ$ (see Figure 2). □

The properties of Ω^{w} are essentially determined by its upper-left block:

Lemma 3.6.

(*i*) Ω^{w} *has a unique negative eigenvalue.*
(*ii*) *if* $\Omega^{\mathrm{w}}\boldsymbol{q} = 0$ *then* $\boldsymbol{q} = A\boldsymbol{p} + t$.

4. The stress-flex conjecture

It was shown in [7] that prestress stability of CPFs would follow from the stress-flex orthogonality for the Wachspress stress. It is now understood that this is, most likely, not specific to the Wachspress stress, but a general property of the bilinear pairing between stresses and first-order flexes in CPFs. We discuss this generalization (the *strong stress-flex conjecture*) in **??**. In this section we recall how the stress-flex conjecture for the Wachspress stress implies prestress stability and how it follows from a vector-valued Schläfli formula proven by Schlenker & Souam [18].

4.1. **From the weak stress-flex conjecture to prestress stability.** We recall the form of the stress-flex conjecture resolved here:

Theorem 1.2. *Let $(G,\boldsymbol{p})$ be a coned polytope framework with Wachspress stress $\boldsymbol{\omega}^{\mathrm{w}}$. If $\dot{\boldsymbol{p}}$ is a first-order motion of $(G,\boldsymbol{p})$ with $\dot{p}_\star = 0$, then*

$$\sum_{i\neq\star} \omega^{\mathrm{w}}_{\star i}\dot{p}_i = 0.$$

Below we provide a self-contained proof for how prestress stability of a CPF follows from Theorem 1.2. The argument is essentially from [7]:

Theorem 1.1. *Coned polytope frameworks are prestress stable.*

Proof. Let Ω^{w} be the stress matrix of the Wachspress stress, and $\dot{\boldsymbol{p}}$ some first-order motion with $\dot{p}_\star = 0$. Following (2.5) it suffices to show (i) that $\mathrm{tr}(\dot{\boldsymbol{p}}\Omega^{\mathrm{w}}\dot{\boldsymbol{p}}^\top) \geq 0$ and (ii) $\mathrm{tr}(\dot{\boldsymbol{p}}\Omega^{\mathrm{w}}\dot{\boldsymbol{p}}^\top) = 0$ only if $\dot{\boldsymbol{p}}$ is trivial.

For (i) let $\boldsymbol{e}_\star \in \mathbb{R}^V$ be the standard unit vector with 1-entry at the cone vertex $\star$. Then the following two identities hold:

$$\begin{aligned} \boldsymbol{e}_\star^\top \Omega^{\mathrm{w}} \boldsymbol{e}_\star &= \Omega^{\mathrm{w}}_{\star\star} \overset{3.5}{=} -\mathrm{vol}(P^\circ - p_\star) < 0, \\ \boldsymbol{e}_\star^\top \Omega^{\mathrm{w}} \dot{\boldsymbol{p}}^\top &= \sum_{i,j} \Omega^{\mathrm{w}}_{ij} e_{\star i}\dot{p}_j = \sum_j \Omega^{\mathrm{w}}_{\star j}\dot{p}_j = \sum_{i\neq\star} \omega^{\mathrm{w}}_{\star i}\dot{p}_i \overset{1.2}{=} 0. \end{aligned} \tag{4.1}$$

Note that it is the last equality where we use stress-flex orthogonality.

Let now $(\dot{\boldsymbol{p}})_k \in \mathbb{R}^V$ denote the k-th row of $\dot{\boldsymbol{p}}$. Then (4.1) states that $\boldsymbol{e}_\star$ and $(\dot{\boldsymbol{p}})_k$ are orthogonal *w.r.t.* the quadratic form induced by Ω^{w}. If $(\dot{\boldsymbol{p}})_k\Omega^{\mathrm{w}}(\dot{\boldsymbol{p}})_k^\top < 0$, then $\boldsymbol{e}_\star$ and $(\dot{\boldsymbol{p}})_k$ would span a 2-dimensional subspace of $\mathbb{R}^V$ on which Ω^{w} is negative definite. But by Lemma 3.6 (i) Ω^{w} has at most one negative eigenvalue. This would be a contradiction. Hence

$$(\dot{\boldsymbol{p}})_k\Omega^{\mathrm{w}}(\dot{\boldsymbol{p}})_k^\top \geq 0 \quad\Longrightarrow\quad \mathrm{tr}(\dot{\boldsymbol{p}}\Omega^{\mathrm{w}}\dot{\boldsymbol{p}}^\top) = \sum_k (\dot{\boldsymbol{p}})_k\Omega^{\mathrm{w}}(\dot{\boldsymbol{p}})_k^\top \geq 0, \tag{4.2}$$

which proves (i).

For (ii), suppose that $\dot{\boldsymbol{p}}\Omega^{\mathrm{w}}\dot{\boldsymbol{p}}^\top = 0$. We need to show that $\dot{\boldsymbol{p}}$ is trivial. In [7] this is argued using affine flexes and ruled surfaces. The subsequent argument is a reformulation thereof that requires no knowledge of these concepts.

First, using (4.2) we can conclude $(\dot{\boldsymbol{p}})_k\Omega^{\mathrm{w}}(\dot{\boldsymbol{p}})_k^\top = 0$ for all k. Since Ω^{w} is positive semi-definite on the span of all $(\dot{\boldsymbol{p}})_k$, we also conclude $\Omega^{\mathrm{w}}(\dot{\boldsymbol{p}})_k = 0$. Lemma 3.6 (ii) then gives us $\dot{\boldsymbol{p}} = A\boldsymbol{p} + t$ for some matrix $A \in \mathbb{R}^{d\times d}$ and vector $t \in \mathbb{R}^d$. The first-order flex condition (2.2) becomes

$$0 = \langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle = (p_i - p_j)^\top A (p_i - p_j), \quad \text{for all } ij \in E. \tag{4.3}$$

In other words, the quadratic form $Q(x) := x^\top Ax$ vanishes on all edge directions of $(G,\boldsymbol{p})$. It now suffices to show $Q = 0$: this is equivalent to A being skew-symmetric and $\dot{\boldsymbol{p}} = A\boldsymbol{p} + t$ being trivial according to (2.3).

In the remainder of the proof we show that for each face $\sigma \subseteq P$, Q vanishes on $E_\sigma := \mathrm{span}(p_i - p_\star \mid p_i \in \sigma)$. The claim $Q = 0$ then follows from $E_\sigma = \mathbb{R}^d$ whenever σ is a facet of P. We proceed by induction on the dimension of the face σ.

Case $\dim(\sigma) = 0$*:* the face σ is a vertex p_i, and Q vanishes on $p_i - p_\star$ by (4.3).

Case $\dim(\sigma) = 1$*:* the face σ is some edge between vertices p_i and p_j. The vectors $p_i - p_\star, p_j - p_\star$ and $p_i - p_j$ are three coplanar edge directions of $(G,\boldsymbol{p})$ (their span is E_σ), and since $p_\star \in \mathrm{int}(P)$, any two are linearly independent. The restriction $Q|_{E_\sigma}$

has degree at most two, but vanishes on three different 1-dimensional subspaces. Hence Q vanishes on all of E_σ.

Case $\dim(\sigma) \geq 2$*:* then there are at least three $(\delta-1)$-dimensional faces $\tau_1, \tau_2, \tau_3 \subset \sigma$. The restriction $Q|_{E_\sigma}$ has degree at most two but, by induction hypothesis, vanishes on three distinct $(\delta-1)$-dimensional subspace $E_{\tau_i} \subset E_\sigma$. Hence Q vanishes on all of E_σ. □

Remark 4.1. The Wachspress stress has the property that it is strictly positive on the cone edges and strictly negative on the polytope edges. Hence, the proof of Theorem 1.1 actually shows something stronger: a CPF is not only prestress stable as a bar-joint framework, but is also prestress stable as a *tensegrity framework* with struts for the polytope edges and cables for the cone edges. In contrast to bars, which must stay of a fixed length during a motion, struts are allowed to get longer (but not shorter), and cables are allowed to get shorter (but not longer). This has also been mentioned in [7]. For the second-order theory of tensegrities, see [8].

4.2. **The vector-valued Schläfli formula and a proof of the stress-flex conjecture.** In this part we show that, in the case of the Wachspress stress, the stress-flex conjecture is a special case of a generalized Schläfli formula proven by Schlenker and Souam [18]. We first recall the essential notions.

From here on, let P^t be a 1-parameter family of polytopes, all of which have the same number of facets $F_1, ..., F_n$. We say that P^t is continuous (resp. differentiable) if both the facet normals n_i^t and facet volumes V_i^t change continuously (resp. differentiably) in t. As before, we drop the superscript t if we evaluate at $t = 0$. We write $\dot{n}_i$ and $\dot{V}_i$ to denote the derivative of n_i^t and V_i^t at $t = 0$ respectively.

Schlenker and Souam proved the following *vector-valued Schläfli formula*:

Theorem 4.2 ([18, Theorem 1.11])**.**

$$\sum_{\sigma_1} \dot{V}_{\sigma_1} n_{\sigma_1} = \sum_{\sigma_2} \dot{\theta}_{\sigma_2} V_{\sigma_2} b_{\sigma_2}.$$

The first sum is over facets σ_1, the second sum is over ridges σ_2, θ_{σ_2} denotes the dihedral angle at σ_2, and $b_{\sigma_2} \in \mathbb{R}^d$ is the barycenter of the facet σ_2.

The relevant special case of Theorem 4.2 can be motivated naturally. Consider the first variation of the Minkowski balancing property (*cf.* Theorem 3.1):

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i n_i V_i = 0 \quad \Longrightarrow \quad \sum_i \dot{n}_i V_i + \sum_i n_i \dot{V}_i = 0 \tag{4.4}$$

It turns out that if the dihedral angles of P^t do not change in first order, then both summands on left side of (4.4) vanish individually:

Corollary 4.3. *If $\dot{\theta}_{ij} = 0$ for all facets F_i and F_j that are adjacent at $t = 0$, then*

$$\sum_i n_i \dot{V}_i = \sum_i \dot{n}_i V_i = 0.$$

Proof. Set $\dot{\theta}_{\sigma_2} = 0$ in Theorem 4.2. □

A second self-contained proof is given in Section 5. The stress-flex conjecture for $\boldsymbol{\omega}^{\mathrm{W}}$ can be seen as a dual formulation of Corollary 4.3. We recall the statement:

Theorem 1.2. *Let $(G, \boldsymbol{p})$ be a coned polytope frameworks with Wachspress stress $\boldsymbol{\omega}^{\mathrm{W}}$. If $\dot{\boldsymbol{p}}$ is a first-order motion of $(G, \boldsymbol{p})$ with $\dot{p}_\star = 0$, then*

$$\sum_{i \neq \star} \omega^{\mathrm{W}}_{\star i} \dot{p}_i = 0.$$

Proof. By translation we may also assume $p_\star = 0$. Recall that there is a differentiable 1-parameters family $\boldsymbol{p}^t$ of embeddings with $\boldsymbol{p}^0 = \boldsymbol{p}$ and $\dot{\boldsymbol{p}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}^t|_{t=0}$. Since $\dot{\boldsymbol{p}}$ is a first-order flex, it preserves bar lengths in first-order. That is

$$0 = \tfrac{\mathrm{d}}{\mathrm{d}t}\|p_i - p_\star\|^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}\|p_i\|^2. \tag{4.5}$$

$$0 = \tfrac{\mathrm{d}}{\mathrm{d}t}\|p_i - p_j\|^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}(\|p_i\|^2 - 2\langle p_i, p_j\rangle + \|p_j\|^2) \overset{(4.5)}{=} -2\tfrac{\mathrm{d}}{\mathrm{d}t}\langle p_i, p_j\rangle. \tag{4.6}$$

Now consider the following 1-parameter family $P^t \subset \mathbb{R}^d$ of polytopes:

$$P^t := \big\{x \in \mathbb{R}^d \mid \langle p_i^t, x\rangle \leq 1 \text{ for all } i \in \{1, ..., n\}\big\}.$$

For $t = 0$ this is precisely the polar dual P°. The number of facets of P^t stays constant within a neighborhood of $t = 0$, and facet normals n_i^t and volumes V_i^t are differentiable in t. In particular, $\dot{n}_i = \dot{p}_i/\|p_i\|$ because $\|p_i\|$ is constant in first order by (4.5). Moreover, the dihedral angles $\theta_{ij} = \arccos\langle n_i, n_j\rangle = \arccos(\langle p_i, p_j\rangle/\|p_i\|\|p_j\|)$ are also constant in first-order by (4.5) and (4.6).

Then for the Wachspress stress $\boldsymbol{\omega}^{\mathrm{W}}$ holds

$$\sum_i \omega^{\mathrm{W}}_{\star i}\dot{p}_i = \sum_i \frac{\mathrm{vol}(F_i)}{\|p_i\|}\dot{p}_i = \sum_i \mathrm{vol}(F_i)\frac{\dot{p}_i}{\|p_i\|} = \sum_i V_i \dot{n}_i = 0, \tag{4.7}$$

where the last equality is Corollary 4.3. ∎

5. A proof of Theorem 1.4

In this section we give a direct discrete-geometric proof of Theorem 1.4. First, recall some notation from Section 1.2. We have a convex polytope $P \subset \mathbb{R}^d$. For faces $\sigma_k \subset \sigma_{k+1}$ of P, $n^{\sigma_{k+1}}_{\sigma_k}$ is the unit normal vector from σ_{k+1} towards σ_k. For faces $\sigma_k \subset \sigma_{k+2}$, $\theta^{\sigma_{k+2}}_{\sigma_k}$ is the exterior dihedral angle of σ_k in σ_{k+2}. For a face σ, V_σ is its volume and b_σ is its barycenter. Given an infinitesimal deformation of facets of P, recall that in Theorem 1.4 we need to show

$$\sum_{\sigma_{d-1}} \dot{V}_{\sigma_{d-1}} n^{\sigma_d}_{\sigma_{d-1}} = \sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} b_{\sigma_{d-2}}\ . \tag{5.1}$$

Denote the origin by $o \in \mathbb{R}^d$. For a face σ, let o_σ be the orthogonal projection from o to the affine span of σ. If necessary, we slightly perturb o so that for any faces $\sigma \neq \tau$, o_σ does not coincide with o_τ. For faces $\sigma \subset \tau$, we denote by h^τ_σ the distance from o_σ to o_τ. However, when σ has codimension one in τ, we consider h^τ_σ as the oriented distance, *i.e.,* taken with the minus sign if o_τ and τ belong to the different sides from the affine span of σ in the affine span of τ.

We need a few preliminary computations for a proof of Theorem 1.4. First, consider three faces $\sigma_k \subset \sigma_{k+1} \subset \sigma_{k+2}$. Denote by α the angle $\measuredangle o_{\sigma_{k+1}} o_{\sigma_{k+2}} o_{\sigma_k}$, taken with the minus sign if $h^{\sigma_{k+1}}_{\sigma_k}$ and $h^{\sigma_{k+2}}_{\sigma_{k+1}}$ have the opposite signs. Note that since o_{k+1} does not coincide with o_k and with o_{k+2}, we get either $0 < \alpha < \pi/2$ or $-\pi/2 < \alpha < 0$. We have

$$h^{\sigma_{k+1}}_{\sigma_k} = h^{\sigma_{k+2}}_{\sigma_{k+1}} \tan\alpha\ .$$

Differentiating it, we obtain

$$\dot h_{\sigma_k}^{\sigma_{k+1}} = \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} \tan\alpha + h_{\sigma_{k+1}}^{\sigma_{k+2}} \frac{\dot\alpha}{\cos^2\alpha} \ .$$

We have

$$\cos^2\alpha = \left(\frac{h_{\sigma_{k+1}}^{\sigma_{k+2}}}{h_{\sigma_k}^{\sigma_{k+2}}}\right)^2 \ .$$

From the last two equations, we get

$$\dot h_{\sigma_k}^{\sigma_{k+1}} = \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} \tan\alpha + \frac{\left(h_{\sigma_k}^{\sigma_{k+2}}\right)^2}{h_{\sigma_{k+1}}^{\sigma_{k+2}}}\dot\alpha \ .$$

Multiplying by $h_{\sigma_{k+1}}^{\sigma_{k+2}}$ and using $h_{\sigma_{k+1}}^{\sigma_{k+2}}\tan\alpha = h_{\sigma_k}^{\sigma_{k+1}}$ yields

$$\dot h_{\sigma_k}^{\sigma_{k+1}} h_{\sigma_{k+1}}^{\sigma_{k+2}} = h_{\sigma_k}^{\sigma_{k+1}} \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} + \left(h_{\sigma_k}^{\sigma_{k+2}}\right)^2 \dot\alpha \ . \tag{5.2}$$

Suppose that both $h_{\sigma_k}^{\sigma_{k+1}}$ and $h_{\sigma_{k+1}}^{\sigma_{k+2}}$ are positive. We differentiate the Pythagorean theorem for the triangle $o_{\sigma_{k+1}}o_{\sigma_{k+2}}o_{\sigma_k}$ and obtain

$$\dot h_{\sigma_k}^{\sigma_{k+1}} \sin\alpha + \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} \cos\alpha = \dot h_{\sigma_k}^{\sigma_{k+2}}.$$

Let n be the unit normal vector from $o_{\sigma_{k+2}}$ towards o_{σ_k} and $n^\perp$ be the unit normal vector from the triangle $o_{\sigma_{k+1}}o_{\sigma_{k+2}}o_{\sigma_k}$ towards $o_{\sigma_k}o_{\sigma_{k+2}}$. From the last two equations, we compute

$$\begin{aligned}\dot\alpha h_{\sigma_k}^{\sigma_{k+2}} n = (\dot h_{\sigma_k}^{\sigma_{k+1}}\cos\alpha - \dot h_{\sigma_{k+1}}^{\sigma_{k+2}}\sin\alpha)(n_{\sigma_k}^{\sigma_{k+1}}\sin\alpha + n_{\sigma_{k+1}}^{\sigma_{k+2}}\cos\alpha) = \\ = \dot h_{\sigma_k}^{\sigma_{k+1}} n_{\sigma_{k+1}}^{\sigma_{k+2}} - \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+1}} + \\ + (\dot h_{\sigma_k}^{\sigma_{k+1}}\sin\alpha + \dot h_{\sigma_{k+1}}^{\sigma_{k+2}}\cos\alpha)(n_{\sigma_k}^{\sigma_{k+1}}\cos\alpha - n_{\sigma_{k+1}}^{\sigma_{k+2}}\sin\alpha) = \\ = \dot h_{\sigma_k}^{\sigma_{k+1}} n_{\sigma_{k+1}}^{\sigma_{k+2}} - \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+1}} + \dot h_{\sigma_k}^{\sigma_{k+2}} n^\perp \ .\end{aligned} \tag{5.3}$$

One can check that for the other possible signs of $h_{\sigma_k}^{\sigma_{k+1}}$ and $h_{\sigma_{k+1}}^{\sigma_{k+2}}$, for the same left-hand side of (5.3), the right-hand side either remains the same or becomes $\dot h_{\sigma_k}^{\sigma_{k+1}} n_{\sigma_{k+1}}^{\sigma_{k+2}} - \dot h_{\sigma_{k+1}}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+1}} - \dot h_{\sigma_k}^{\sigma_{k+2}} n^\perp$.

Second, pick two faces $\sigma_k \subset \sigma_{k+2}$. Note that then there exist exactly two $(k+1)$-faces σ_{k+1} and σ_{k+1}^* such that $\sigma_k \subset \sigma_{k+1} \subset \sigma_{k+2}$ and $\sigma_k \subset \sigma_{k+1}^* \subset \sigma_{k+2}$. Define α as above, and similarly define α^* for the triangle $o_{\sigma_{k+1}^*}o_{\sigma_{k+2}}o_{\sigma_k}$. Note that either both pairs of numbers $h_{\sigma_k}^{\sigma_{k+1}}$, $h_{\sigma_{k+1}}^{\sigma_{k+2}}$ and $h_{\sigma_k}^{\sigma_{k+1}^*}$, $h_{\sigma_{k+1}^*}^{\sigma_{k+2}}$ have the same sign or they both have the opposite signs. This means that both α and α^* have the same sign. Observe that, depending on the sign, either

$$\alpha + \alpha^* = \theta_{\sigma_k}^{\sigma_{k+2}} \qquad \text{or} \qquad -\alpha - \alpha^* = \pi - \theta_{\sigma_k}^{\sigma_{k+2}} \ .$$

In either case,

$$\dot\alpha + \dot\alpha^* = \dot\theta_{\sigma_k}^{\sigma_{k+2}} \ . \tag{5.4}$$

Let n be the unit normal vector from $o_{\sigma_{k+2}}$ towards o_{σ_k}, $n^\perp$ be the unit normal vector from the triangle $o_{\sigma_{k+1}}o_{\sigma_{k+2}}o_{\sigma_k}$ towards $o_{\sigma_k}o_{\sigma_{k+2}}$ and $n^{\perp *}$ be the unit normal vector from the triangle $o_{\sigma_{k+1}^*}o_{\sigma_{k+2}}o_{\sigma_k}$ towards $o_{\sigma_k}o_{\sigma_{k+2}}$. We have $n^\perp = -n^{\perp *}$. Note also that the term $\dot h_{\sigma_k}^{\sigma_{k+2}} n^\perp$ appears in (5.3) for the triangle $o_{\sigma_{k+1}}o_{\sigma_{k+2}}o_{\sigma_k}$

with the same sign as the term $\dot{h}_{\sigma_k}^{\sigma_{k+2}} n^{\perp *}$ appears in such equality for the triangle $o_{\sigma^*_{k+1}} o_{\sigma_{k+2}} o_{\sigma_k}$. From this, (5.4) and (5.3), we deduce

$$\begin{aligned}(5.5)\quad \dot{\theta}_{\sigma_k}^{\sigma_{k+2}} h_{\sigma_k}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+2}} &= (\dot{\alpha} + \dot{\alpha}^*) h_{\sigma_k}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+2}} = \\ &= \dot{h}_{\sigma_k}^{\sigma_{k+1}} n_{\sigma_{k+1}}^{\sigma_{k+2}} - \dot{h}_{\sigma_{k+1}}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma_{k+1}} + \dot{h}_{\sigma_k}^{\sigma^*_{k+1}} n_{\sigma^*_{k+1}}^{\sigma_{k+2}} - \dot{h}_{\sigma^*_{k+1}}^{\sigma_{k+2}} n_{\sigma_k}^{\sigma^*_{k+1}} .\end{aligned}$$

Third, pick two faces $\sigma_k \subset \sigma_{k+3}$. Consider the spherical link of σ_k in σ_{k+3}. Theorem 1.7 implies that

$$(5.6)\qquad \sum_{\substack{\sigma_{k+2}:\\ \sigma_k \subset \sigma_{k+2} \subset \sigma_{k+3}}} \dot{\theta}_{\sigma_k}^{\sigma_{k+2}} n_{\sigma_{k+2}}^{\sigma_{k+3}} = \sum_{\substack{\sigma_{k+1}:\\ \sigma_k \subset \sigma_{k+1} \subset \sigma_{k+3}}} \dot{\theta}_{\sigma_{k+1}}^{\sigma_{k+3}} n_{\sigma_k}^{\sigma_{k+1}} .$$

Now we are ready to begin our proof.

Proof of Theorem 1.4. Let f be a full flag of faces of P. We denote its respective elements by $f_0, \dots, f_d$, according to dimension. We write for short $h_{f_k} := h_{f_k}^{f_{k+1}}$ and $n_{f_k} := n_{f_k}^{f_{k+1}}$. Denote by α_{f_k} the angle $\measuredangle o_{f_{k-1}} o_{f_k} o_{f_{k-2}}$, taken with the minus sign if $h_{f_{k-1}}$ and $h_{f_{k-2}}$ have opposite signs. Denote by V_f the oriented volume of f, *i.e.*, $V_f := \frac{1}{d!} h_{f_0} \dots h_{f_{d-1}}$.

We say that a flag having a face in each dimension up to and including k is a *k-order flag*. We denote such a flag by f^k and denote its elements by $f^k_0, \dots, f^k_k$. The conventions for full flags apply to flags of any order.

Given a face σ_k, its volume V_{σ_k} is

$$(5.7)\qquad V_{\sigma_k} = \sum_{f^k : f^k_k = \sigma_k} V_{f^k} = \frac{1}{k!} \sum_{f^k : f^k_k = \sigma_k} h_{f^k_0} \dots h_{f^k_{k-1}} .$$

In particular,

$$V_{\sigma_{d-1}} = \frac{1}{(d-1)!} \sum_{f : f_{d-1} = \sigma_{d-1}} h_{f_0} h_{f_1} \dots h_{f_{d-2}} .$$

A variation of this gives

$$(5.8)\qquad \dot{V}_{\sigma_{d-1}} = \frac{1}{(d-1)!} \sum_{f : f_{d-1} = \sigma_{d-1}} \sum_{k=0}^{d-2} h_{f_0} \dots \dot{h}_{f_k} \dots h_{f_{d-2}} .$$

For a full flag f and $k : 0 \le k \le d-3$, equation (5.2) yields

$$\dot{h}_{f_k} h_{f_{k+1}} = h_{f_k} \dot{h}_{f_{k+1}} + \left(h_{f_k}^{f_{k+2}} \right)^2 \dot{\alpha}_{f_{k+2}} .$$

Multiplying this equation by the other orthonumbers, we obtain

$$\begin{aligned} &h_{f_0} \dots h_{f_{k-1}} \dot{h}_{f_k} h_{f_{k+1}} \dots h_{f_{d-2}} = \\ &\quad h_{f_0} \dots h_{f_k} \dot{h}_{f_{k+1}} h_{f_{k+2}} \dots h_{f_{d-2}} + h_{f_0} \dots h_{f_{k-1}} \left(h_{f_k}^{f_{k+2}} \right)^2 \dot{\alpha}_{f_{k+2}} h_{f_{k+2}} \dots h_{f_{d-2}} .\end{aligned}$$

Using it in (5.8), we get

$$\begin{aligned}(5.9)\quad \dot{V}_{\sigma_{d-1}} &= \frac{1}{(d-2)!} \sum_{f : f_{d-1} = \sigma_{d-1}} h_{f_0} \dots h_{f_{d-3}} \dot{h}_{f_{d-2}} + \\ &+ \frac{1}{(d-1)!} \sum_{f : f_{d-1} = \sigma_{d-1}} \sum_{k=0}^{d-3} (k+1) h_{f_0} \dots h_{f_{k-1}} \left(h_{f_k}^{f_{k+2}} \right)^2 \dot{\alpha}_{f_{k+2}} h_{f_{k+2}} \dots h_{f_{d-2}} .\end{aligned}$$

Note that we of course mean that in the second sum for $k = 0$ the corresponding term does not have the part $h_{f_0} \dots h_{f_{k-1}}$ and for $k = d-3$ the corresponding term does not have the part $h_{f_{k+2}} \dots h_{f_{d-2}}$.

Define

$$T := \sum_{\sigma_{d-1}} \dot{V}_{\sigma_{d-1}} n^{\sigma_d}_{\sigma_{d-1}} \ .$$

Substitute there (5.9) and obtain

$$T = \sum_{\sigma_{d-2} \subset \sigma_{d-1}} V_{\sigma_{d-2}} \dot{h}^{\sigma_{d-1}}_{\sigma_{d-2}} n^{\sigma_d}_{\sigma_{d-1}} + \\ + \frac{1}{(d-1)!} \sum_f \sum_{k=0}^{d-3} (k+1) h_{f_0} \dots h_{f_{k-1}} \left(h^{f_{k+2}}_{f_k}\right)^2 \dot{\alpha}_{f_{k+2}} h_{f_{k+2}} \dots h_{f_{d-2}} n_{f_{d-1}} \ .$$

Denote the summands

$$T_1 := \sum_{\sigma_{d-2} \subset \sigma_{d-1}} V_{\sigma_{d-2}} \dot{h}^{\sigma_{d-1}}_{\sigma_{d-2}} n^{\sigma_d}_{\sigma_{d-1}} \ ,$$

$$T_2 := \frac{1}{(d-1)!} \sum_f \sum_{k=0}^{d-3} (k+1) h_{f_0} \dots h_{f_{k-1}} \left(h^{f_{k+2}}_{f_k}\right)^2 \dot{\alpha}_{f_{k+2}} h_{f_{k+2}} \dots h_{f_{d-2}} n_{f_{d-1}} \ .$$

Let us deal with T_1. Given any σ_{d-2}, there exist exactly two faces σ_{d-1} and σ^*_{d-1} such that $\sigma_{d-2} \subset \sigma_{d-1}$ and $\sigma_{d-2} \subset \sigma^*_{d-1}$. Coupling the summands this way and using (5.5), we obtain

$$T_1 = \sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} o_{\sigma_{d-2}} + \sum_{\sigma_{d-2} \subset \sigma_{d-1}} V_{\sigma_{d-2}} n^{\sigma_{d-1}}_{\sigma_{d-2}} \dot{h}^{\sigma_d}_{\sigma_{d-1}} \ .$$

The second sum is zero by applying the Minkowski theorem, Theorem 1.5, to every σ_{d-1}. Hence,

$$T_1 = \sum_{\sigma_{d-2}} \dot{\theta}^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} o_{\sigma_{d-2}} \ . \tag{5.10}$$

Now it remains to handle T_2. For every $k : 0 \leq k \leq d-2$ and every full flag f, there exists a unique other full flag f^* coinciding with f in all dimensions but $k+1$. By applying the Minkowski theorem, Theorem 1.5, to the quadrilateral $o_{f_k} o_{f_{k+1}} o_{f^*_{k+1}} o_{f_{k+2}}$, we obtain

$$h_{f_k} n_{f_{k+1}} + h_{f^*_k} n_{f^*_{k+1}} = n_{f_k} h_{f_{k+1}} + n_{f^*_k} h_{f^*_{k+1}} \ .$$

From such couplings of flags, we deduce

$$T_2 = \frac{1}{(d-1)!} \sum_f \sum_{k=0}^{d-3} (k+1) h_{f_0} h_{f_1} \dots h_{f_{k-1}} \left(h^{f_{k+2}}_{f_k}\right)^2 \dot{\alpha}_{f_{k+2}} n_{f_{k+2}} h_{f_{k+3}} \dots h_{f_{d-1}} \ . \tag{5.11}$$

The Schläfli formula, Theorem 1.3, says that for every σ_{k+2} we have

$$\sum_{\sigma_k : \sigma_k \subset \sigma_{k+2}} V_{\sigma_k} \dot{\theta}^{\sigma_{k+2}}_{\sigma_k} = 0 \ .$$

Together with (5.7) and (5.4), it imples that for every σ_{k+2} we obtain

$$\sum_{f^{k+2} : f^{k+2}_{k+2} = \sigma_{k+2}} h_{f^{k+2}_0} \dots h_{f^{k+2}_{k-1}} \dot{\alpha}_{f^{k+2}_{k+2}} = 0 \ .$$

From the Pythagorean theorem for the triangle $o_{f_k}o_{f_{k+2}}o_{f_{k+3}}$, we get

$$\left(h_{f_k}^{f_{k+2}}\right)^2=\left(h_{f_k}^{f_{k+3}}\right)^2-\left(h_{f_{k+2}}^{f_{k+3}}\right)^2 .$$

Using the last two equations in (5.11), we have

$$T_2=\frac{1}{(d-1)!}\sum_f\sum_{k=0}^{d-3}(k+1)h_{f_0}\dots h_{f_{k-1}}\left(h_{f_k}^{f_{k+3}}\right)^2\dot\alpha_{f_{k+2}}n_{f_{k+2}}h_{f_{k+3}}\dots h_{f_{d-1}} . \tag{5.12}$$

Fix $k: 0\le k\le d-3$ and fix a flag F that has a face in every dimension but $k+1$ and $k+2$. Notice that (5.6) and the couplings of flags imply that

$$\sum_{f:F\subset f}\dot\alpha_{f_{k+2}}n_{f_{k+2}}=\sum_{f:F\subset f}n_{f_k}\dot\alpha_{f_{k+3}} . \tag{5.13}$$

We apply (5.13) in (5.12) and obtain

$$T_2=\frac{1}{(d-1)!}\sum_f\sum_{k=0}^{d-3}(k+1)h_{f_0}\dots h_{f_{k-1}}\left(h_{f_k}^{f_{k+3}}\right)^2 n_{f_k}\dot\alpha_{f_{k+3}}h_{f_{k+3}}\dots h_{f_{d-1}} . \tag{5.14}$$

The Minkowski theorem, Theorem 1.5, says that for every σ_{k+1} we have

$$\sum_{f^{k+1}:f^{k+1}_{k+1}=\sigma_{k+1}}h_{f_0^{k+1}}\dots h_{f_{k-1}^{k+1}}n_{f_k}=0 .$$

From the Pythagorean theorem for the triangle $o_{f_k}o_{f_{k+1}}o_{f_{k+3}}$, we get

$$\left(h_{f_k}^{f_{k+3}}\right)^2=(h_{f_k})^2+\left(h_{f_{k+1}}^{f_{k+3}}\right)^2 .$$

Using the last two equations in (5.14), we get

$$T_2=\frac{1}{(d-1)!}\sum_f\sum_{k=0}^{d-3}(k+1)h_{f_0}\dots h_{f_{k-1}}(h_{f_k})^2 n_{f_k}\dot\alpha_{f_{k+3}}h_{f_{k+3}}\dots h_{f_{d-1}} . \tag{5.15}$$

For a full flag f and $k: 0\le k\le d-1$, we have $\langle n_{f_k},o_{f_{k+1}}\rangle=0$. For any $l\le k$, o_{f_l} belongs to the affine span of f_k. Hence, it belongs to the hyperplane orthogonal to n_{f_k} at the oriented distance h_{f_k} from $o_{f_{k+1}}$. This implies that for any $l\le k$,

$$h_{f_k}=\langle n_{f_k},o_{f_l}-o_{f_{k+1}}\rangle=\langle n_{f_k},o_{f_l}\rangle . \tag{5.16}$$

Now fix $k: 0\le k\le d-4$ and fix a flag F that has a face in every dimension but $k+2$ and $k+3$. Combining (5.16) and (5.13), we obtain

$$\sum_{f:F\subset f}\dot\alpha_{f_{k+3}}h_{f_{k+3}}=\sum_{f:F\subset f}\langle\dot\alpha_{f_{k+3}}n_{f_{k+3}},o_{f_{k+1}}\rangle=$$
$$=\sum_{f:F\subset f}\langle n_{f_{k+1}}\dot\alpha_{f_{k+4}},o_{f_{k+1}}\rangle=\sum_{f:F\subset f}h_{f_{k+1}}\dot\alpha_{f_{k+4}} .$$

Note that here we use that f_{k+1} is the same for all f containing F.

We apply it in (5.15) and get

$$T_2=\frac{1}{(d-1)!}\sum_f\sum_{k=0}^{d-3}(k+1)h_{f_0}\dots h_{f_{k-1}}(h_{f_k})^2 n_{f_k}h_{f_{k+1}}\dots h_{f_{d-3}}\dot\alpha_{f_d} . \tag{5.17}$$

Now we unpack the right-hand side of (5.1). For $l : 0 \leq l \leq d$ and an l-order flag f^l, denote by b_{f^l} the barycenter of f^l, *i.e.*, $b_{f^l} := \frac{1}{l+1}\sum_{k=0}^{l} o_{f^l_k}$. Note that for $k : 0 \leq k \leq l-1$,

$$o_{f^l_k} - o_{f^l_l} = \sum_{m=k}^{l-1} (o_{f^l_m} - o_{f^l_{m+1}}) = \sum_{m=k}^{l-1} h_{f^l_m} n_{f^l_m} \ .$$

Hence,

$$b_{f^l} - o_{f^l_l} = \frac{1}{l+1}\sum_{k=0}^{l-1}(k+1) h_{f^l_k} n_{f^l_k} \ .$$

Let Q_1 and Q_2 be two full-dimensional polytopes in $\mathbb{R}^k$ with disjoint interiors, V_1 and V_2 be their volumes, b_1 and b_2 be their barycenters and b be the barycenter of their union. By the basic property of barycenters,

$$(V_1 + V_2) b = V_1 b_1 + V_2 b_2 \ .$$

In turn, this means that for every face σ_l, we have

$$\begin{aligned} V_{\sigma_l}(b_{\sigma_l} - o_{\sigma_l}) &= \sum_{f^l : f^l_l = \sigma_l} V_{f^l}(b_{f^l} - o_{\sigma_l}) = \\ &= \frac{1}{l!}\sum_{f^l : f^l_l = \sigma_l} h_{f^l_0} \ldots h_{f^l_{l-1}} \frac{1}{l+1}\sum_{k=0}^{l-1}(k+1) h_{f^l_k} n_{f^l_k} = \\ &= \frac{1}{(l+1)!}\sum_{f^l : f^l_l = \sigma_l}\sum_{k=0}^{l-1}(k+1) h_{f^l_0} \ldots h_{f^l_{k-1}} (h_{f^l_k})^2 n_{f^l_k} h_{f^l_{k+1}} \ldots h_{f^l_{l-1}} \ . \end{aligned}$$

By applying this to every σ_{d-2}, from (5.17), we conclude that

$$T_2 = \sum_{\sigma_{d-2}} \dot\theta^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} (b_{\sigma_{d-2}} - o_{\sigma_{d-2}}) \ .$$

Together with (5.10), this shows that

$$T = T_1 + T_2 = \sum_{\sigma_{d-2}} \dot\theta^{\sigma_d}_{\sigma_{d-2}} V_{\sigma_{d-2}} b_{\sigma_{d-2}} \ ,$$

which finishes the proof. □

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Appendix A. Proof of Theorem 1.7

By an infinitesimal deformation here we call an equivalence class of assignments of tangent vectors to each n_i modulo the Killing fields on $\mathbb{S}^2$. A trivial observation is that an infinitesimal deformation of our configuration is uniquely determined by the infinitesimal changes of the lengths of the segments $n_1n_2, \ldots, n_{k-1}n_k$ (so, all but one) and by the infinitesimal changes of the angles at $n_2, \ldots, n_{k-1}$ (so, all but two).

Let

$$\Phi : \mathbb{R}^3 \to \mathfrak{so}(3)$$

be the cross-product isomorphism, where we perceive $\mathfrak{so}(3)$ as the space of the Killing fields on $\mathbb{S}^2$. Namely, we send $x \in \mathbb{R}^3$ to the vector field $\Phi(x)$ on $\mathbb{S}^2 \subset \mathbb{R}^3$ defined by $\Phi(x)(y) := y \times x$ for $y \in \mathbb{S}^2$, where $\times$ is the Euclidean cross product in $\mathbb{R}^3$. Define a_1 to be the zero Killing field and define

$$a_i := \sum_{j=1}^{i-1} \Phi(\dot{\alpha}_j n_j - \dot{\beta}_j m_j) \ .$$

We claim that the set $a_i(n_i)$ (which is a choice of a tangent vector at each n_i) is a representative of our infinitesimal deformation. Indeed,

$$a_{i+1} - a_i = \Phi(\dot{\alpha}_i n_i - \dot{\beta}_i m_i) \ , \qquad i = 1, \ldots, n-1 \ .$$

By evaluating $a_{i+1} - a_i$ at n_{i+1}, we see that this choice of vectors indeed induces the infinitesimal change $\dot{\beta}_i$ on the length of n_in_{i+1}, $i = 1, \ldots, k-1$. By evaluating $a_{i+1} - a_i$ at n_{i+1} and $a_{i-1} - a_i$ at n_{i-1}, we see that this choice of vectors indeed induces the infinitesimal change $\dot{\alpha}_i$ on the angle $n_{i-1}n_in_{i+1}$, $i = 2, \ldots, k-1$. Hence, it indeed represents our infinitesimal deformation. Now, by evaluating $a_1 - a_k$ and evaluating $\Phi(\dot{\alpha}_k n_k - \dot{\beta}_k m_k)$ at n_1 and n_k, we see that they have the same values at n_1 and n_k, hence

$$a_1 - a_k = \Phi(\dot{\alpha}_k n_k - \dot{\beta}_k m_k) \ .$$

Substituting there $a_1 = 0$ and $a_k = \sum_{j=1}^{k-1} \Phi(\dot{\alpha}_j n_j - \dot{\beta}_j m_j)$, we get

$$\Phi(\sum_{j=1}^{k} \dot{\alpha}_j n_j - \dot{\beta}_j m_j) = 0 \ ,$$

from which the statement of Theorem 1.7 follows.