Energies on Coned Convex Polytopes

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Abstract

We give a streamlined derivation of the main techinical lemma of a paper by Winter using the language of equilibrium stresses.

1 Introduction

Recently a number of rigidity results for the one-skeleton of a convex polytope coned from an interior point have been proven by Winter [7]. One result says that the resulting tensegrity framework is locally rigid. Another result says that the framework is globally rigid, when only allowing for new configurations that arise as the one-skeleton of a (combinatorially equivalent) polytope (in particular the "faces" must remain flat). All of these results follow through a central (technical) Lemma 4.3 in [7]. In what follows we will call this "the coned-polytope lemma". The proof given for this lemma is somewhat involved and perhaps ad-hoc. In this note, we give a streamlined proof of the coned-polytope lemma and describe it using the standard notion of equilibrium stresses from rigidity theory.

In terms of techniques, the results in [7] rely on a result of Izmestiev [4] that generalizes a three-dimensional result of Lovász [6]. This result associates a special n-by-n matrix M to a convex polytope P (with n vertices) in $\mathbb{R}^d$ that contains the origin in its interior. Among other things, M has a single negative eigenvalue. In this note, we add an extra row and column to M to obtain $(n+1)$-by-$(n+1)$ matrix Ω with a single negative eigenvalue. By construction, this matrix will be a stress matrix for the framework of the coned polytope. This stress matrix is then used in our exploration of the coned-polytope lemma.

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2 Definitions

Here we quickly review the basic rigidity definitions that we will be needing. The key reference is [1].

Definition 2.1. *A configuration* $\mathbf{p}$ *of* n *points in* $\mathbb{R}^d$ *is an ordered set of* n *points,* $\mathbf{p}_i \in \mathbb{R}^d$.

Definition 2.2. *A tensegrity framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^d$ *is a configuration* $\mathbf{p}$ *of* n *points* $\mathbb{R}^d$*, and a labeled graph* G *on* n *vertices. The edges of* G *are labeled as "cables", "struts" and "bars". We denote by* $(\bar{G}, \mathbf{p})$ *the associated bar framework, where all of the labels in* G *are changed to be "bars".*

Definition 2.3. *We say that a tensgrity framewok* $(G, \mathbf{q})$ *is dominated by* $(G, \mathbf{p})$ *if the struts only get larger going from* $\mathbf{p}$ *to* $\mathbf{q}$*, the cables only get shorter going from* $\mathbf{p}$ *to* $\mathbf{q}$ *and the bar lenghts remain the same.*

The following definition will not be used but is given for context.

Definition 2.4. *A tensegrity framework $(G, \mathbf{p})$ is (locally) rigid if for every configuration $\mathbf{q}$ of n points in $\mathbb{R}^d$ that is sufficiently close to $\mathbf{p}$*[1] *and such that $\mathbf{q}$ is not congruent to $\mathbf{p}$, we have the property that the Euclidean lengths of the cables are not decreased, the lengths of the struts are not increased and the lengths of the bars are not changed.*

Definition 2.5. *Let $(G, \mathbf{p})$ be a tensegrity with n vertices. An (equilibrium) stress matrix Ω of $(G, \mathbf{p})$ is an n-by-n symmetric matrix with the following properties. $\Omega_{ij} = 0$ when $i \neq j$ and ij is not an edge of G. We have $\Omega\mathbf{p} = 0$, where we think of $\mathbf{p}$ as an n-by-d matrix. We also have $\Omega\mathbf{1} = 0$, where $\mathbf{1}$ is a vector of all ones.*

The stress is called strictly proper if $\Omega_{ij} < 0$ when ij is a cable and $\Omega_{ij} > 0$ when ij is a strut.

Definition 2.6. *Let Ω be an n-by-n symmetric matrix with $\Omega\mathbf{1} = 0$. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. We define the associated stress energy as*

$$E_s(\mathbf{x}) := \mathbf{x}^t \Omega \mathbf{x}$$

This can be expanded out as

$$E_s(\mathbf{x}) = \sum_{i<j} -\Omega_{ij}(x_i - x_j)^2$$

If $\mathbf{q}$ is a configuration of n points in $\mathbb{R}^d$, thought of as an n-by-d matrix, we define

$$E_s(\mathbf{q}) := \mathrm{tr}(\mathbf{q}^t \Omega \mathbf{q}) = \sum_{i<j} -\Omega_{ij}||\mathbf{q}_i - \mathbf{q}_j||^2$$

3 Izmestiev Stress

Let $P \subset \mathbb{R}^d$ be a convex polytope with n vertices, with a full affine span and with the origin in its interior. Let $\mathbf{p}$ be its vertex configuration and let G be the graph of its one-skeleton. Izmestiev [4] constructs an n-by-n symmetric matrix M, that we call the *Izmestiev matrix*, with the following properties:

(i) $M_{ij} = 0$ when $i \neq j$ and ij is not an edge of G.

(ii) $M_{ij} < 0$ when ij is an edge of G.

(iii) $M\mathbf{p} = 0$, where we think of $\mathbf{p}$ as an n-by-d matrix.

(iv) M has rank $n - d$.

(v) M has exactly one negative eigenvalue.

Our goal is to use M to construct an equilibrium stress Ω for the framework $(G^*, \hat{\mathbf{p}})$ obtained by coning the one-skeleton of P over the origin. Let $\alpha^t := -\mathbf{1}^t M$ and $b := -\sum_i \alpha_i = \mathbf{1}^t M \mathbf{1}$. The α_i are also known as the (unnormalized) Wachspress coordinates of the origin with respect to P [7], and so each is greater than 0, and b is negative. We define the $(n+1)$-by-$(n+1)$ matrix in block form as follows:

[1] Formally, there exists an ϵ so that for every $\mathbf{q}$ within this distance to $\mathbf{p}$, the property holds.

$$\Omega := \begin{pmatrix} M & \alpha \\ \alpha^t & b \end{pmatrix}$$

The added row/column are linearly dependent on M, and thus the rank of Ω is also $n-d$, and its nullity is $d+1$. Since M has one negative eigenvalue, and Ω just has an extra 0 eigenvalue, from the eigenvalue interlacing theorem, Ω must also have exactly one negative eigenvalue.

Let $\mathbf{c}$ be placed at the origin and let $\bar{\mathbf{p}} = [\mathbf{p}, \mathbf{c}]$ be the configuration of $n+1$ points in $\mathbb{R}^d$. Then, since $\mathbf{c} = 0$ we have $\Omega\bar{\mathbf{p}} = 0$. By our definition of α and b, we have $\Omega\mathbf{1} = 0$. Let us label the edges of G^* from the polytope as cables and the coned edges as struts. We see that Ω is a strictly proper equilibrium stress for $(G^*, \bar{\mathbf{p}})$. We call this an *Izmestiev stress*.

Finally we note that the space of stresses for a framework does not change under translation in $\mathbb{R}^d$. Thus we can start with P as any convex polytope with n vertices and a full affine span in $\mathbb{R}^d$ and with $\mathbf{c}$ any point in the interior of P. We can create its associated coned tensegrity, and it must have an Izmestiev stress.

4 One Negative Eigenvalue

Rigidity analysis is often based on positivity of some quadratic energy. Our stress matrix Ω has one negative eigenvalue that we need to work around. Our main tool for doing this is the following lemma.

Lemma 4.1. *Let Ω be an $(n+1)$-by-$(n+1)$ real symmetric matrix. Assume the following:*

(i) Ω has exactly one negative eigenvalue.

(ii) $\Omega_{n+1,n+1} < 0$.

If $\hat{\mathbf{x}} \in \mathbb{R}^{n+1}$ has the property that the $(n+1)$st entry of $\Omega\hat{\mathbf{x}}$ equals 0, then $\hat{\mathbf{x}}^t\Omega\hat{\mathbf{x}} \geq 0$.

Proof. Let $\mathbf{e}$ be the $(n+1)$st indicator vector. By assumption $\mathbf{e}^t\Omega\mathbf{e} < 0$, so the vector, $\mathbf{e}$ is in the negative cone of Ω. Meanwhile, since the last entry of $\Omega\hat{\mathbf{x}}$ equals 0, we have $\mathbf{e}^t\Omega\hat{\mathbf{x}} = 0$, i.e., $\hat{\mathbf{x}}$ is Ω-orthogonal to $\mathbf{e}$. Together, since Ω has only a single negative eigenvalue, this implies $\hat{\mathbf{x}}^t\Omega\hat{\mathbf{x}} \geq 0$. □

Similar ideas to Lemma 4.1 are used in [5].

5 Energy Zoo

Starting with an Izmestiev stress Ω we decompose its stress energy in a way which will lead to a streamlined path towards the coned-polytope lemma (see Section 6).

Assume an Izmestiev stress Ω for $(G^*, \bar{\mathbf{p}})$, with $\bar{\mathbf{p}} = [\mathbf{p}; \mathbf{c}]$ the framework of the one skeleton of a convex polytope P, coned over the point $\mathbf{c}$. We assume the stress is normalized to be of the form

$$\Omega = \begin{pmatrix} M & \alpha \\ \alpha^t & -1 \end{pmatrix}$$

Let E_s be the stress energy associated with Ω. Let $\hat{\mathbf{q}} = [\mathbf{q}; \mathbf{q}_{n+1}]$ be a configuration of $n+1$ points. Then we have

$$\begin{aligned} E_s(\hat{\mathbf{q}}) &= \sum_{i<j} -M_{ij}||\mathbf{q}_i - \mathbf{q}_j||^2 - \sum_i \alpha_i ||\mathbf{q}_i - \mathbf{q}_{n+1}||^2 \\ &=: E_x(\mathbf{q}) - E_i(\hat{\mathbf{q}}) \end{aligned}$$

Definition 5.1. *E_x is called the external energy and E_i is called the internal energy.*

Definition 5.2. *Given a configuration $\mathbf{q}$ of n points we define its α-center as $\mathbf{q}_\alpha := \sum_i \alpha_i \mathbf{q}_i$.*

Note that since Ω is an equilibrium stress for $\bar{\mathbf{p}}$, in particular the last row of $\Omega\bar{\mathbf{p}}$ equals 0, and so we see that $\bar{\mathbf{p}} = [\mathbf{p}; \mathbf{p}_\alpha]$.

Lemma 5.3. *Given a framework $(G^*, \hat{\mathbf{q}})$ we can write*

$$\begin{aligned} E_i(\hat{\mathbf{q}}) &= \sum_{i<j} \alpha_i \alpha_j ||\mathbf{q}_i - \mathbf{q}_j||^2 + ||\mathbf{q}_\alpha - \mathbf{q}_{n+1}||^2 \\ &=: E_\alpha(\mathbf{q}) + E_c(\hat{\mathbf{q}}) \end{aligned}$$

Proof. This is a calculation in [7, Equation 2.1] (it relies on our normalization for Ω). ☐

Definition 5.4. *E_α is called the α-energy and E_c is called the cone energy.*

Now we can re-parenthesize as follows

$$\begin{aligned} E_s(\hat{\mathbf{q}}) &= E_x(\mathbf{q}) - (E_\alpha(\mathbf{q}) + E_c(\hat{\mathbf{q}})) \\ &= (E_x(\mathbf{q}) - E_\alpha(\mathbf{q})) - E_c(\hat{\mathbf{q}}) \\ &=: E_p(\mathbf{q}) - E_c(\hat{\mathbf{q}}) \end{aligned}$$

Definition 5.5. *E_p is called the polytope energy.*

Note that all energies are invariant to translation on $\hat{\mathbf{q}}$. External, internal, α, and cone energies are non-negative. Only E_s and E_c depend on the cone placement. The main point here is that we have decomposed E_s as the difference of two energies, where the first energy does not depend on the $(n+1)$st point and the second energy has a particular simple form.

The payoff of this decomposition in the case of an Izemestiev stress is the following lemma:

Lemma 5.6. *E_p is positive semi-definite (PSD). Its kernel consists of the affine images of $\mathbf{p}$.*

Recall that $\mathbf{p}$ is the conguration of the vertices of P used, along with the cone point $\mathbf{c}$, to define Ω and thus E_p.

Proof. Let $\mathbf{q}$ be any configuration of n points. Let $\tilde{\mathbf{q}}$ be the configuration of $n+1$ points with its first n points in agreement with $\mathbf{q}$, and its $(n+1)$st point at $\mathbf{q}_\alpha$. By this choice for $\mathbf{q}_{n+1}$, we have $E_c(\tilde{\mathbf{q}}) = 0$ and so $E_s(\tilde{\mathbf{q}}) = E_p(\mathbf{q})$. Meanwhile each of the d spatial coordinates of $\tilde{\mathbf{q}}$ satisfies the assumption of Lemma 4.1 and so $E_s(\tilde{\mathbf{q}}) \geq 0$. Thus $E_p(\mathbf{q}) \geq 0$.

Next we characterize its kernel. Suppose that $\mathbf{q}$ is an affine image of $\mathbf{p}$, namely for all i, $\mathbf{q}_i = A\mathbf{p}_i + \mathbf{t}$ where A is a d-by-d matrix and $\mathbf{t} \in \mathbb{R}^d$. Then $\mathbf{q}_\alpha = A\mathbf{p}_\alpha + \mathbf{t}$. Defining $\tilde{\mathbf{q}} := [\mathbf{q}, \mathbf{q}_\alpha]$ we see that $\tilde{\mathbf{q}}$ is an affine image of $\bar{\mathbf{p}} = [\mathbf{p}; \mathbf{p}_\alpha]$. Meanwhile since Ω is an equilibrium stress for $\bar{\mathbf{p}}$ we have $E_s(\bar{\mathbf{p}}) = 0$. As an affine transform, we also have $E_s(\tilde{\mathbf{q}}) = 0$. Meanwhile $E_s(\tilde{\mathbf{q}}) = E_p(\mathbf{q})$. Since E_p is PSD, $\mathbf{q}$ must be in its kernel.

Going the other way, suppose that $\mathbf{q}$ is in the kernel of E_p. Letting $\tilde{\mathbf{q}} := [\mathbf{q}, \mathbf{q}_\alpha]$, we have $E_s(\tilde{\mathbf{q}}) = 0$. Using Lemma 4.1, we know that $\tilde{\mathbf{q}}$ lies in a linear space on which Ω is PSD. Since $E_s(\tilde{\mathbf{q}}) = 0$ it must be in the kernel of Ω. Thus $\tilde{\mathbf{q}}$ is an affine image of $\bar{\mathbf{p}}$ and $\mathbf{q}$ is an affine image of $\mathbf{p}$. ☐

6 The Coned-Polytope Lemma

Using Lemma 5.6, we can now directly prove (a restated version of) the coned-polytope lemma.

Lemma 6.1. *[7, Lemma 4.3] Let P be a convex polytope with vertex configuration $\mathbf{p}$. Let $\bar{\mathbf{p}} := [\mathbf{p};\mathbf{c}]$ where $\mathbf{c}$ is in the interior of P. Let $(G^*, \bar{\mathbf{p}})$ be the framework of the coned skeleton of P. The framework $(G^*, \bar{\mathbf{p}})$ gives rise to an Izmestiev stress Ω, various energies, E_s, E_p and in particular the vector α and the energy E_c.*

Let $\hat{\mathbf{p}} = [\mathbf{p};\mathbf{p}_{n+1}]$ be a configuration with $\mathbf{p}$ as above and where $\mathbf{p}_{n+1}$ is any cone point. Let $\hat{\mathbf{q}} = [\mathbf{q};\mathbf{q}_{n+1}]$ be any configuration.

Suppose that $(G^*, \hat{\mathbf{q}})$ is dominated by $(G^*, \hat{\mathbf{p}})$. And suppose that $E_c(\hat{\mathbf{p}}) \geq E_c(\hat{\mathbf{q}})$. Then $\hat{\mathbf{q}} \cong \hat{\mathbf{p}}$.

Proof. Since $E_s(\bar{\mathbf{p}}) = 0$ and $\mathbf{c} = \mathbf{p}_\alpha$, we have $E_s(\bar{\mathbf{p}}) = E_p(\mathbf{p}) = 0$.

$$\begin{array}{ccccc} E_s(\hat{\mathbf{p}}) & = & E_p(\mathbf{p}) & - & E_c(\hat{\mathbf{p}}) \\ & = & 0 & & \\ \geq & & \leq & & \leq \\ E_s(\hat{\mathbf{q}}) & = & E_p(\mathbf{q}) & - & E_c(\hat{\mathbf{q}}) \end{array}$$

The first inequality comes from the assumed dominance. The second inequality uses the fact that E_p is PSD (Lemma 5.6). The third inequality is explicitly assumed. Thus all of the inequalities are equalities. Since $E_p(\mathbf{q}) = 0$, $\mathbf{q}$ is an affine image of $\mathbf{p}$ (Lemma 5.6). Meanwhile, since $E_s(\hat{\mathbf{p}}) = E_s(\hat{\mathbf{q}})$ and $\hat{\mathbf{p}}$ dominates $\hat{\mathbf{q}}$, we must have all of the lengths equal.

To finally conclude that $\mathbf{p}$ and $\hat{\mathbf{q}}$ are congruent, we need a statement about affine flexes that we included in the appendix: using Lemma A.1, and the fact that P is a convex polytope with a full affine span, we see that $\hat{\mathbf{q}}$ must be congruent to $\hat{\mathbf{p}}$. □

The name of game in [7] is to flip this around: one starts with $(G^*, \hat{\mathbf{p}})$ with $\hat{\mathbf{p}} = [\mathbf{p};\mathbf{p}_{n+1}]$, a framework of a convex polytope P coned over the point $\mathbf{p}_{n+1}$, and $\hat{\mathbf{q}}$, a framework of the same graph that is dominated by $\hat{\mathbf{p}}$. Then one sees if one can find a point $\mathbf{c}$ inside of P, so that, (under the resulting stress/energies/α, defined using the Izmestiev equilibrium stress for $\bar{\mathbf{p}} := [\mathbf{p};\mathbf{c}]$) we have $E_c(\hat{\mathbf{p}}) \geq E_c(\hat{\mathbf{q}})$. If this can be done, then the proposition can be applied to show congruence of $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$.

A Affine Flexes

Lemma A.1. *Let $(G, \mathbf{p})$ be a framework on n vertices in $\mathbb{R}^d$. Let $(G, \mathbf{q})$ be a framework with $\mathbf{q}_i = L\mathbf{p}_i + \mathbf{t}$, where L is a d-by-d matrix and $\mathbf{t} \in \mathbb{R}^d$. Let $(G^*, \hat{\mathbf{p}})$ and $(G^*, \hat{\mathbf{q}})$ be the respective frameworks, coned over the origin.*

Suppose that $(G^*, \hat{\mathbf{p}})$ and $(G^*, \hat{\mathbf{q}})$ are equivalent but not congruent. Then the vertices and the supporting lines of the edges of $(G, \mathbf{p})$ lie on an (possibly) inhomogeoneous quadratic surface in $\mathbb{R}^d$.

Moreover, such a $(G, \mathbf{p})$ cannot have d vertices in general affine position, each with a neighborhood in $(G, \mathbf{p})$ with a full affine span. In particular it cannot be the skeleton of a convex polytope.

Proof. Let $\hat{\mathbf{r}} := (1/2)(\hat{\mathbf{p}} + \hat{\mathbf{q}})$ and $\hat{\mathbf{r}}' := (1/2)(\hat{\mathbf{p}} - \hat{\mathbf{q}})$. Since $(G^*, \hat{\mathbf{p}})$ and $(G^*, \hat{\mathbf{q}})$ are equivalent, from the averaging principle [2, Theorem 13], $\hat{\mathbf{r}}'$ is a infinitesimal flex for $(G^*, \hat{\mathbf{r}})$. For $i \in [n]$, we have $\hat{\mathbf{r}}_i = 1/2(\mathbf{p}_i + L\mathbf{p}_i + \mathbf{t})$ and $\hat{\mathbf{r}}'_i = 1/2(\mathbf{p}_i - L\mathbf{p}_i - \mathbf{t})$. At the cone, we have $\hat{\mathbf{r}}_{n+1} = 0$ and $\hat{\mathbf{r}}'_{n+1} = 0$.

Since $\hat{\mathbf{r}}'$ is a flex, just looking at the cone-edge flex condition, $\hat{\mathbf{r}}_i \cdot \hat{\mathbf{r}}'_i = 0$, we get, after a calculation, for $i \in [n]$

$$\mathbf{p}_i^t(I - L^tL)\mathbf{p}_i - 2\mathbf{p}_i^tL\mathbf{t} - \mathbf{t}^t\mathbf{t} = 0$$

This places the $\mathbf{p}_i$ on a quadratic surface in $\mathbb{R}^d$. (The equation would only be trivial when L is an orthogonal matrix and $\mathbf{t} = 0$, making $(G^*, \hat{\mathbf{p}})$ and $(G^*, \hat{\mathbf{q}})$ congruent.)

Looking at the flex condition on the edges ij in G, we get, after a calculation

$$\mathbf{e}_{ij}^t(I - L^tL)\mathbf{e}_{ij} = 0$$

where $\mathbf{e}_{ij} := \mathbf{p}_i - \mathbf{p}_j$. This gives us three points from each edge ij of G (the points $\mathbf{p}_i$ and $\mathbf{p}_j$, along with the point $\mathbf{e}_{ij}$ at infinity) on the quadric, and thus the supporting lines of the edges of $(G, \mathbf{p})$ are on the quadric [3, Lemma 2.5].

The last statment about affine spanning vertices is [3, Proposision 3.4]. □

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