– Wachspress Coordinates –

A bridge between algebra, geometry and combinatorics

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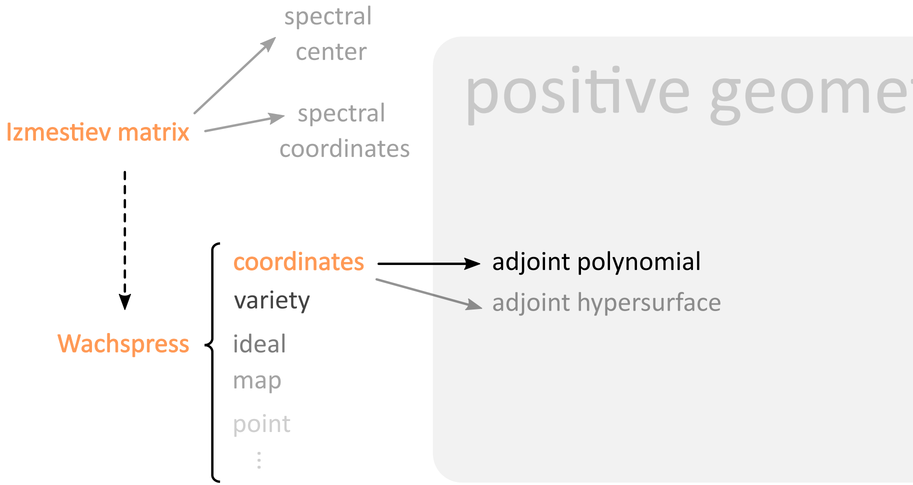
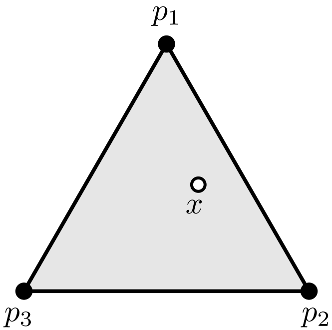
The Wachspress family

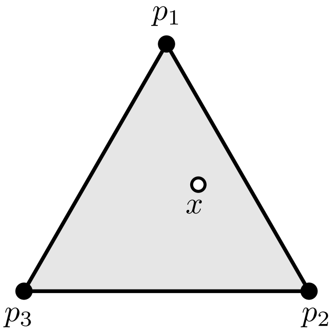


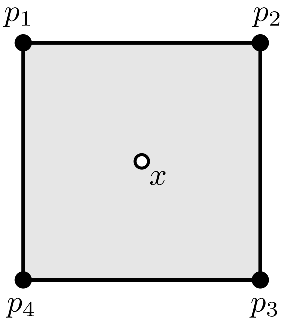
Barycentric coordinates



$$x = \sum_i \alpha_i(x) p_i, \qquad \boldsymbol{\alpha} \in \Delta_n := \{\boldsymbol{\alpha} \in [0,1]^n \mid \alpha_1 + \cdots + \alpha_n = 1\}.$$

Barycentric coordinates for polytopes (??)





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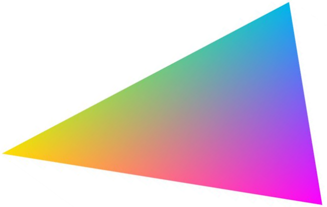
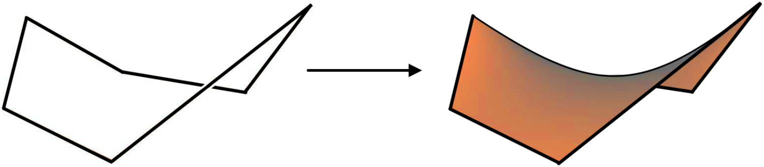
Application: interpolation

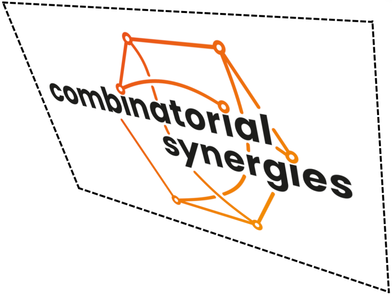


$$\text{DATA}(x) = \sum_i \alpha_i(x)\,\text{DATA}(p_i)$$

- computer graphics
- finite element analysis
- ...



Application: image warping



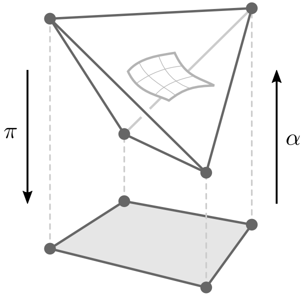
Generalized barycentric coordinates

$\{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1)\}$

$\downarrow$

Generalized barycentric coordinates (GBCs): $\alpha : P \to \Delta_n$ satisfy

$$\sum_i \alpha_i(x) p_i = x \quad \text{(linear precision)}$$



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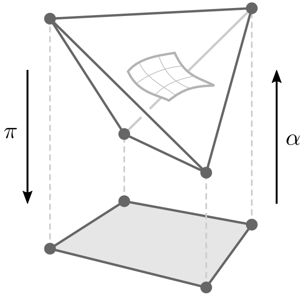
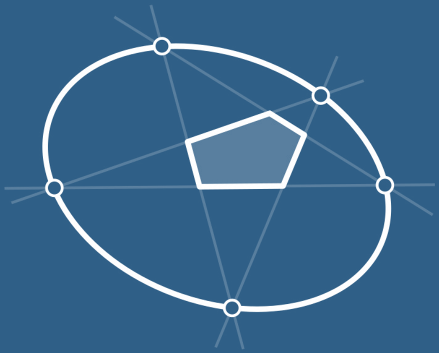
There are ...

- harmonic coordinates,
- mean value coordinates,
- ...
- **Wachspress coordinates**
  (Wachspress, 1975; Warren, 1996)



The many faces of Wachspress coordinates



Wachspress coordinates as rational GBCs

- There do not always exist *polynomial* GBCs. (Wachspress)
- Wachspress constructed *rational GBCs*:

$$\alpha_i(x) = \frac{\mathrm{p}_i(x)}{\mathrm{q}(x)}$$

where $\mathrm{q}(x) = \sum_i \mathrm{p}_i(x)$ ... adjoint polynomial

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Idea: if $x \in \mathsf{face}_k$ but $p_i \notin \mathsf{face}_k$, then $\alpha_i(x) = 0$:

$$\mathrm{p}_i(x) = \beta_i(x) \prod_{k:i \notin \mathsf{face}_k} H_k(x).$$

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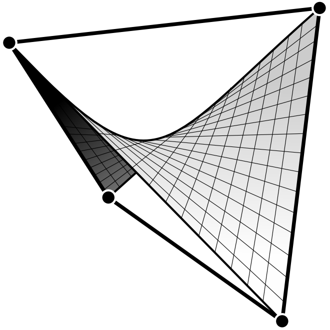
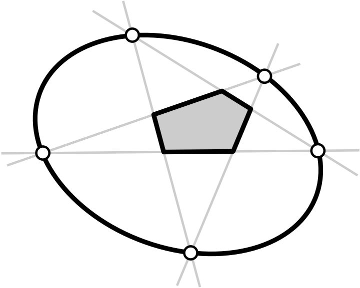
Theorem. (Warren)

The Wachspress coordinates are the unique rational GBCs of lowest possible degree. *degree = #facets − dim*

Wachspress in algebraic geometry

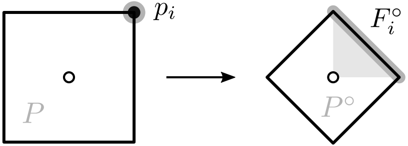
- **Wachspress variety** $V(P) := \mathrm{im}(\alpha) \subseteq \Delta_n$
- **Wachspress ideal** $I(P)$
- **adjoint hypersurface** ... vanishing set of adjoint poynomial





Wachspress from cone volumes (Ju et al., 2005)

polar dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \le 1 \text{ for all } i \in V(G_P)\}$.



$$\alpha_i := \frac{\mathrm{vol}(F_i^\circ)}{\|p_i\| \,\mathrm{vol}(P^\circ)}$$

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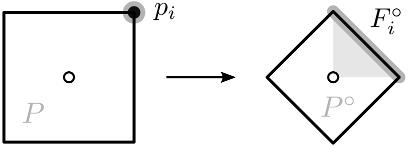


$$\alpha_i := \frac{\mathrm{vol}(F_i^\circ)}{\|p_i\| \, \mathrm{vol}(P^\circ)}$$

$\sum_i \mathrm{vol}(\text{face}_i) \cdot \text{normal}_i = 0$

$$\sum_i \alpha_i p_i = \frac{1}{\mathrm{vol}(P^\circ)} \cdot \sum_i \mathrm{vol}(F_i^\circ) \frac{p_i}{\|p_i\|} = 0.$$

Wachspress from algebraic statistics

(Kohn, Shapiro, Sturmfels; 2020)

- let μ_P be the uniform measure on a polytope $P^\circ \subset \mathbb{R}^d$.
- compute its moments:

$$m_I := \int_{P^\circ} \boldsymbol{x}^I \,\mathrm{d}\boldsymbol{x} = \int_{P^\circ} x_1^{i_1} \cdots x_d^{i_d} \,\mathrm{d}\boldsymbol{x}, \quad I = \{i_1 < \cdots i_d\} \in \mathbb{N}^d$$

- compute the moment generating function:

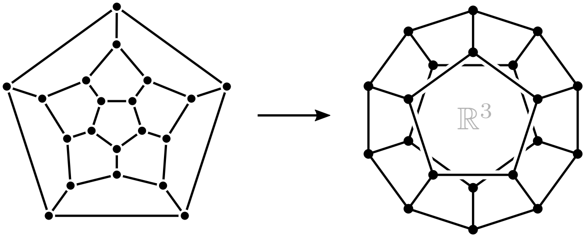
$$\sum_{I \in \mathbb{N}^d} \frac{\left(\sum I + d\right)!}{I!} m_I \boldsymbol{t}^I.$$

$\implies$ this is a rational function whose numerator is the adjoint polynomial of P.

Wachspress from spectral graph theory

$$\theta \in \mathrm{Spec}(A) \implies u_1, ..., u_d \in \mathrm{Eig}_\theta(A)$$

$$\implies \begin{bmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{bmatrix} = \begin{bmatrix} — & p_1 & — \\ & \vdots & \\ — & p_n & — \end{bmatrix} \in \mathbb{R}^{n \times d}$$



$$\mathrm{Spec}(A) = \{\, 3^1, \sqrt{5}^3, 1^5, 0^4, (-2)^4, (-\sqrt{5})^3 \,\}$$

Wachspress from spectral graph theory

Theorem. (Izmestiev, 2010)

A polytope skeleton is a spectral embedding of the edge graph w.r.t. suitable edge and vertex weights.

weight matrix $M \in \mathbb{R}^{n\times n}$ ... **Izmestiev matrix** of P

Applications:

- rigidity of polyhedral frameworks
- relations between polytopal symmetries and edge graph symmetries
- progress on the Hirsch conjecture
  (Narayanan, Shah, Srivastava; 2022)

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$$\alpha_i := \sum_j M_{ij}$$ (W., 2023)

Izmestiev's Theorem

Theorem. (Izmestiev, 2007)

The Izmestiev matrix satisfies

(i) $M_{ij} > 0$ *whenever* $ij \in E$,

(ii) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E$,

(iii) $\dim \ker(M) = d$,

(iv) $MX_P = 0$, *where* $X_P^\top = (p_1, ..., p_n) \in \mathbb{R}^{d \times n}$,

(v) M *has a single positive eigenvalue of multiplicity 1.* *(Lorentzian)*

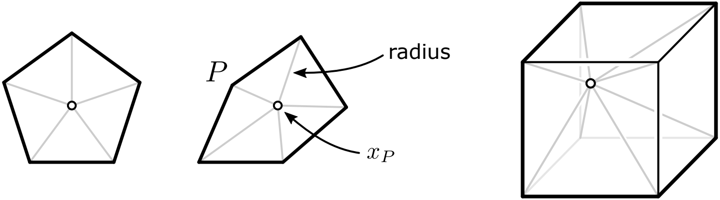
Consequences:

- Defines a function $P \ni x \mapsto \theta_1(x) > 0$ Where are the extremal values?
- M has a unique strictly positive eigenvector $z \in \mathbb{R}^n_+$ (to θ_2):

  $\implies$ defines GBC's $P \ni x \mapsto z(x)$ $=:$ **spectral coordinates**

Pointed polytopes

:= polytope $P \subset \mathbb{R}^d$ + point $x_P \in \mathrm{int}(P)$



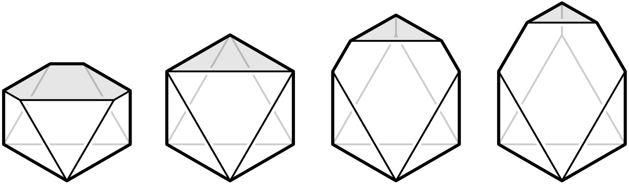
We can speak of

- *the* polar dual of a pointed polytope
- *the* Wachspress coordinates of a pointed polytope
- *the* Izmestiev matrix of a pointed polytope
- ...

Wachspress from variation of volume

$$P^\circ(\mathbf{c}) := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq c_i \text{ for all } i \in V(G_P)\}.$$

where $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{R}^n$.



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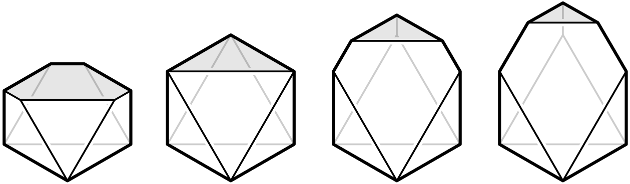
where $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{R}^n$. Expand $\mathrm{vol}(P^\circ(\mathbf{c}))$ at $\mathbf{c} = \mathbf{1}$:

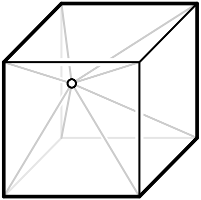
$$\mathrm{vol}(P^\circ(\mathbf{c})) = \mathrm{vol}(P^\circ) + \langle \tilde\alpha, \mathbf{c} - \mathbf{1} \rangle + \tfrac{1}{2}(\mathbf{c} - \mathbf{1})^\top \tilde M (\mathbf{c} - \mathbf{1}) + \cdots$$

↑ Wachspress coordinates

↑ Izmestiev matrix



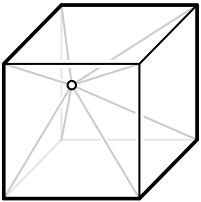
Wachspress from rigidity theory



stress ... $\boldsymbol{\omega} : E \to \mathbb{R}$

$$\forall i \in V: \quad \sum_{j:ij \in E} \omega_{ij}(p_j - p_i) = 0$$

Wachspress from rigidity theory



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Lemma.

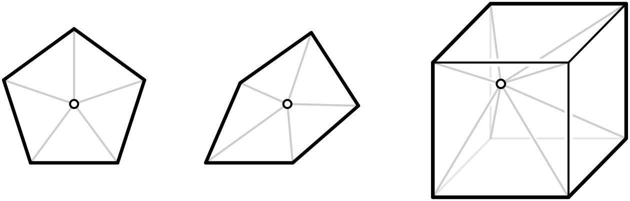
If P is simple, then its framework has a unique non-zero stress and

(i) *stresses on the radial bars (i.e. $\omega_{0i}, i \in V$) are Wachspress coordinates*

(ii) *stresses on the edge bars (i.e. $\omega_{ij}, ij \in E$) are Izmestiev matrix entries.*

Rigidity and reconstruction

Application: rigidity and reconstruction



Theorem. (W., 2023)

A pointed polytope is uniquely determined (up to affine transformation) by its edge graph, edge lengths and Wachspress coordinates.

... across all dimensions and all combinatorial types!

Question: is there a relation to the *log-Minkowski problem*?

Application: rigidity and reconstruction

Conjecture

A pointed polytope P is uniquely determined (up to isometry) by its edge-graph, edge lengths and radii.

Implications:

- reconstruction of matroids from base exchange graph
- strengthening of Kirszbraun theorem
- symmetries of a polytope are encoded in edge lengths and radii.
- ...

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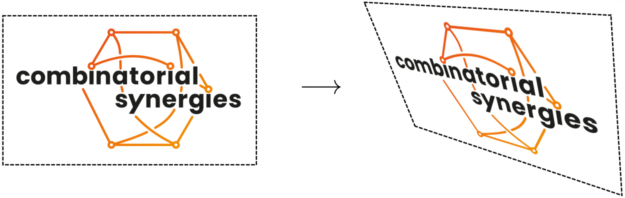
Using the **Izmestiev matrix** one can verify the conjecture if... (W., 2023)

- P, Q centrally symmetric
- $P \approx Q$ (Hausdorff metric)
- $P \simeq Q$ (combinatorially equivalent)

The Wachspress map $\phi\colon P \to Q$

The **Wachspress map** $\phi : P \to Q$ maps

$$x \in P \quad \longmapsto \quad \alpha(x) \in \Delta_n \quad \longmapsto \quad \phi(x) := \sum_i \alpha_i(x) q_i \in Q$$



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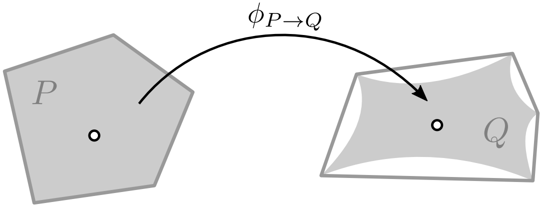

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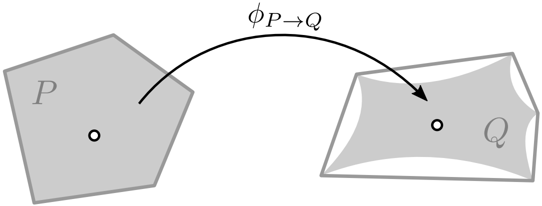

Question: Is there always a point $x \in \operatorname{int}(P)$ with $\|\phi(x) - x_Q\| \le \|x - x_P\|$?

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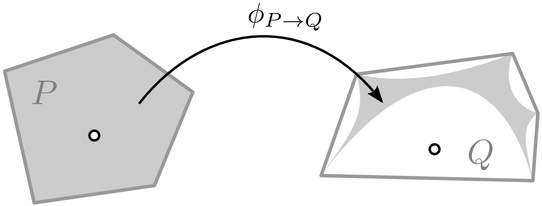

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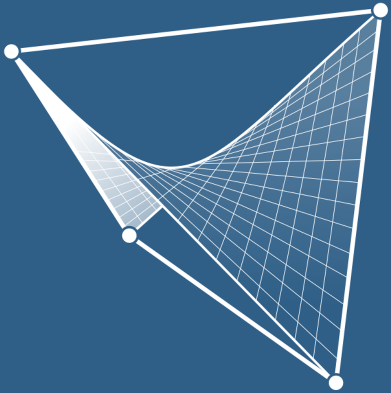
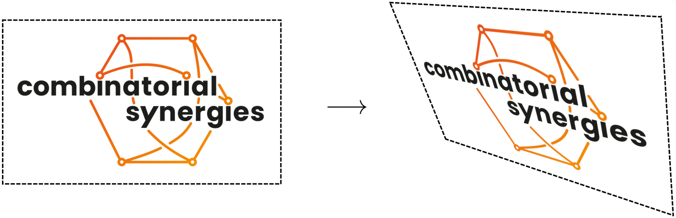
Understanding the variety is key



Injectivity of the Wachspress map



Wachspress map: $x \in P \longmapsto \alpha^P(x) \in \Delta_n \longmapsto \sum_i \alpha_i^P(x)\, q_i \in Q$

Conjecture.

The Wachspress map is injective.

Injectivity of the Wachspress map

Conjecture.

The Wachspress map is injective.

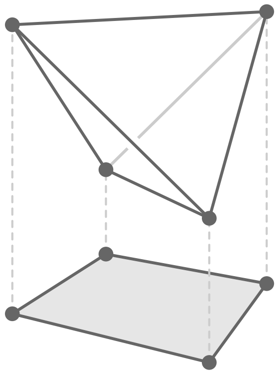
- true in dimension $d = 2$. (Floater, Kosinka; 2008)
- open in dimension $d \geq 3$.
- other commonly used GBCs are not injective!

- Understanding injectivity = understanding secant directions of $V(P)$

Wachspress ideals vs. Stanley-Reisner ideals

P ... **simplicial** polytope

Wachspress variety
↓
$$V(P) \cap \partial\Delta_n \simeq \partial P$$



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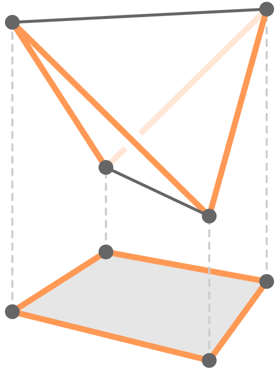

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Wachspress variety
$\downarrow$

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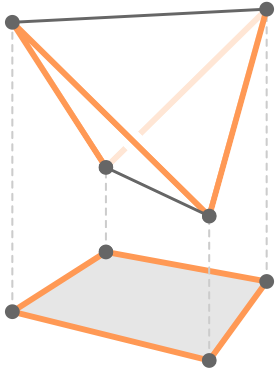
Observation:

- $I(P) = \langle f_1, f_2, ...\rangle$
- the monomials of f_i correspond to the non-faces of P.



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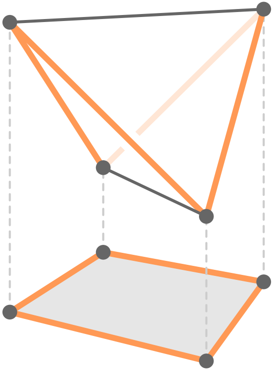

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some relation
Wachspress ideal ↓ $\sim$ Stanley-Reisner ideal

Wachspress ideals vs. Stanley-Reisner ideals

Theorem. (Irving, Schenck, 2013)

For polygons ($d = 2$) holds

- *the initial ideal of the Wachspress ideal (using graded lex order) is given by the Stanley-Reisner ideal.*
- *the Wachspress variety is*
  - *arithmetically Cohen-Macaulay,*
  - *of Castelnuovo-Mumford regularity two.*

Question: How does this generalize to $d \geq 3$?

Deciding polytopality of simplicial spheres

→ NP hard! in NP?

$S \subset \partial\Delta_n$ ... d-dimensional **simplicial sphere**

Task: find a variety $V \subset \Delta_n$ so that ...

- $V \cap \partial\Delta_n = S$.

  → $I(V)$ is generated by polynomials using minimal non-faces.
- the graph of a rational function of degree $m - d$.
- smooth inside of Δ_n.
- ...

If **No**, then S is not polytopal!

Thank you.

M. Winter, *"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"* (2023)