– The Stress-Flex Conjecture –

A riddle of rigidity in coned polytopes

Martin Winter

in joined work with Robert Connelly, Steven Gortler and Louis Theran

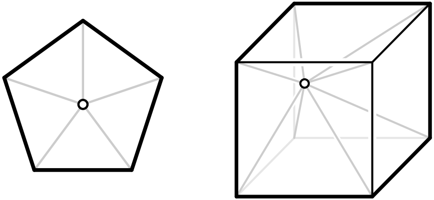
TU Berlin

12. September, 2024

Coned polytope frameworks

A **coned polytope framework** (CPF) consists of

- the 1-skeleton of a polytope
- an interior point (the cone point)
- edges between the cone point and polytope vertices.



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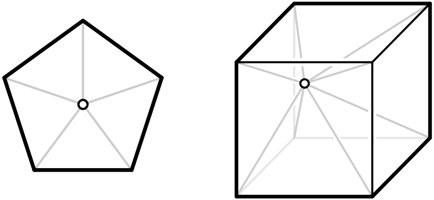
CPFs provide a link between

- rigidity theory,
- polytope theory,
- convex geomety,
- Wachspress geometry.



The stress-flex conjecture (briefly)

- There are objects called **first-order flexes**.
- There are objects called **stresses**.
- The flexes and stresses of CPFs appear to be "orthogonal" to each other, even though one would not expect this.

The stress-flex conjecture (Connelly, Gortler, Theran, W.)

Let $(G_P^\star, \boldsymbol{p})$ be a coned polytope framework. For any choice of

- *first-order flex $\dot{\boldsymbol{p}}\colon V \to \mathbb{R}^d$,* *with $\dot{p}_\star = 0$*
- *stress $\boldsymbol{\omega}\colon E \to \mathbb{R}$,* *we write $\omega_i := \omega_{i\star}$*

holds

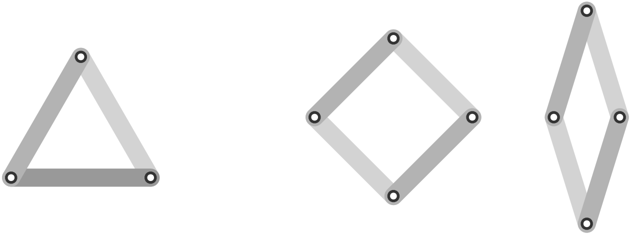
$$\sum_{i \neq \star} \omega_i \dot{p}_i = 0. \quad \longleftarrow \quad \textit{stress-flex orthogonality}$$

- The stress-flex orthogonality appears to holds in much greater generality

A Hands-on Crash Course in Rigidity Theory

Frameworks

$= \text{ graph } G = (V, E) \; + \; \text{embedding } \boldsymbol{p}\colon V \to \mathbb{R}^d$



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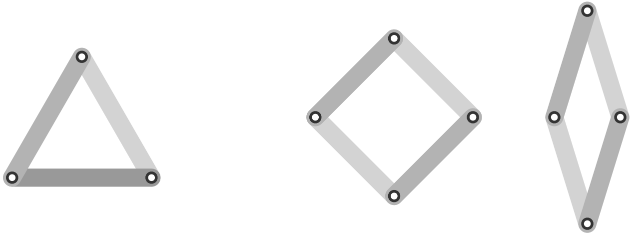

Typical questions:

- Is it rigid?
- If *Yes*, how much rigid? $\rightarrow$ first-order rigid, globally rigid, generically rigid, ...
- If *No*, how does it flex? $\rightarrow$ realization spaces

Rigidity

A **framework** is a pair $(G, \boldsymbol{p})$ with a graph G and an embedding $\boldsymbol{p}\colon V \to \mathbb{R}^d$.

- $(G, \boldsymbol{p})$ and $(G, \boldsymbol{q})$ are **congruent** if   i.e. are the same up to orientation

$$\|p_i - p_j\| = \|q_i - q_j\|, \quad \text{for all } i, j \in V.$$

- $(G, \boldsymbol{p})$ and $(G, \boldsymbol{q})$ are **equivalent** if   i.e. have the same edge lengths

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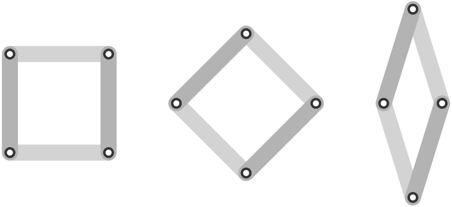

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- A **flex** of $(G, \boldsymbol{p})$ is a continuous family $(G, \boldsymbol{p}^t), t \in [0, 1]$ of pairwise equivalent frameworks with $\boldsymbol{p}^0 = \boldsymbol{p}$.
- A flex is **trivial** if all $(G, \boldsymbol{p}^t)$ are pairwise congruent.
- $(G, \boldsymbol{p})$ is **flexible** if there is a non-trivial flex. Otherwise $(G, \boldsymbol{p})$ is **rigid**.

Hands-on: CPFs

Question: are CPFs rigid?

- Determining whether a framework $(G, \boldsymbol{p})$ is rigid is ... not easy.

  One would need to understand the realization space

  $$\textsc{Real}(G, \boldsymbol{p}) := \big\{(G, \boldsymbol{q}) \mid \|p_j - p_i\| = \|q_j - q_i\| \;\forall ij \in E\big\}.$$

  Is it a single point? Is it discrete? ...

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Theorem. (W., 2023)

Coned polytope frameworks are rigid.

$(G_P^\star, \boldsymbol{p})$ is the unique minimizer of the *polytope energy*

$$E(\boldsymbol{q}) := \sum_{i,j} \omega_i \omega_j \|q_i - q_j\|^2.$$

where $\boldsymbol{\omega}$ are the Wachspress coordinates of the cone point in P.

Hands-on: CPFs

Conjecture.

A CPF is uniquely determined by its graph and edge lengths.

Attention: this is a strong statement!

- we do not input the polytope's combinatorics.
- we do not input the polytope's dimension.

Theorem. (W., 2023)

The conjecture is true

- *locally at a given CPF.*
- *for centrally symmetric CPFs.*
- *for given combinatorial type.*

Are we done ... ?

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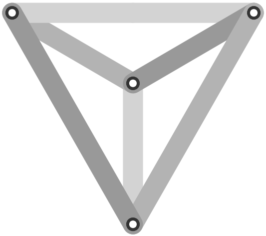
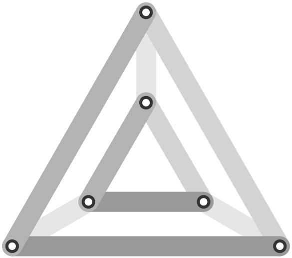
Not quite. We also want to know "how much rigid" are CPFs?





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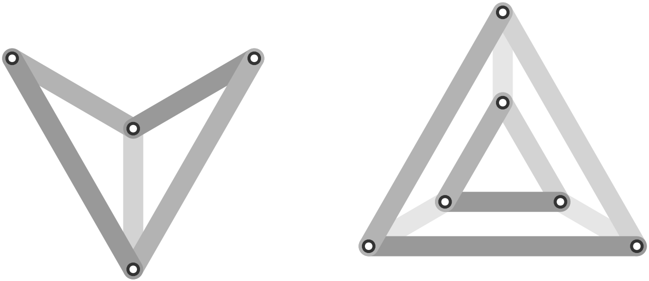

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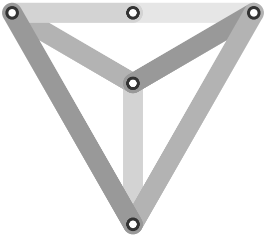

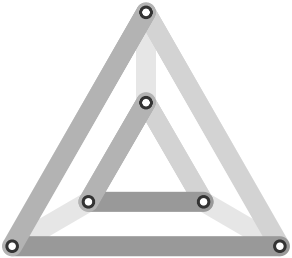

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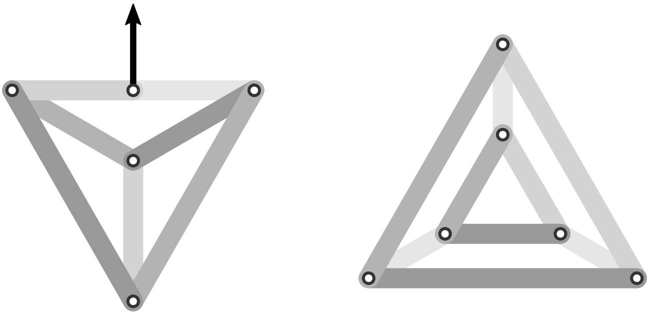

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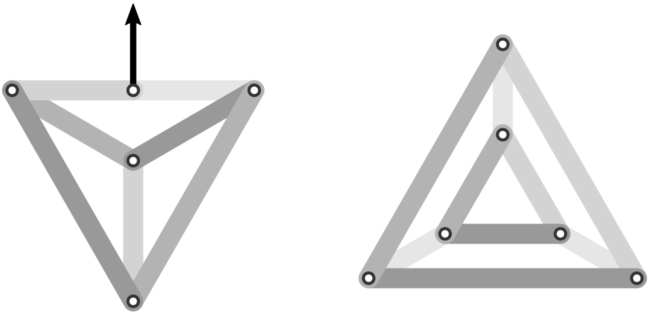

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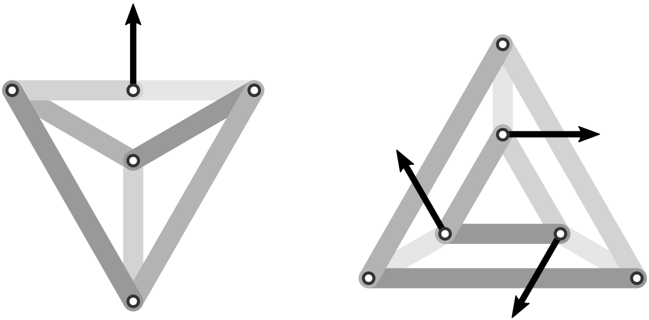

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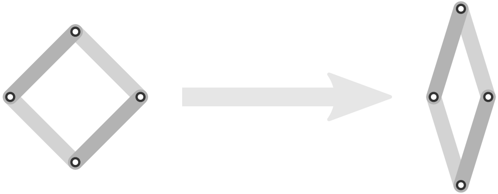

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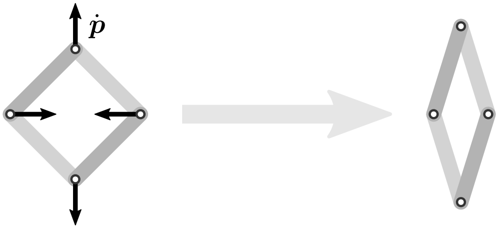

First-order rigidity = infinitesimal rigidity

$$\|p_j^t - p_i^t\| = \text{const} \quad\implies\quad \langle p_j^t - p_i^t, \dot p_j^t - \dot p_i^t\rangle = 0$$

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A **first-order flex** of $(G, \boldsymbol{p})$ is a map $\dot{\boldsymbol{p}}\colon V \to \mathbb{R}^d$ with

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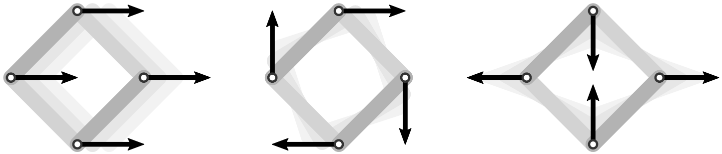

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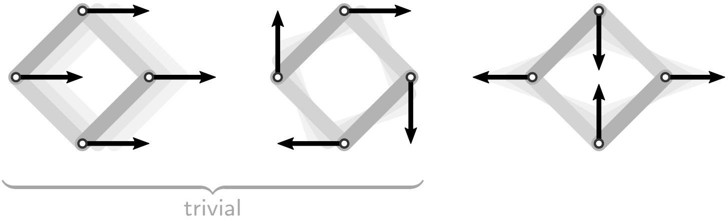
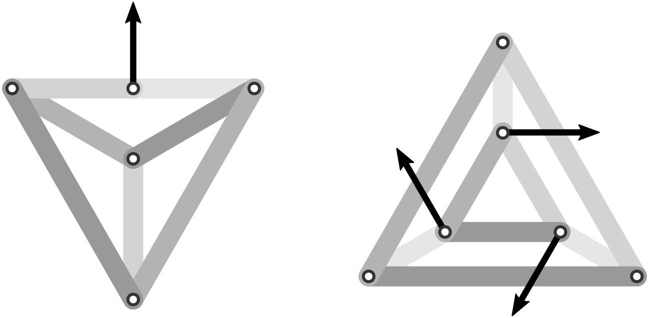
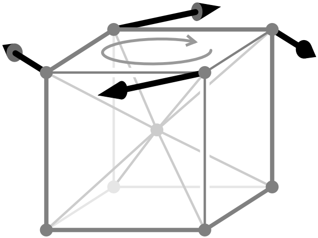


First-order theory isn't perfect



Hands-on: CPFs



simplicial polytope:

- always first-order rigid (coned or not). (Dehn, 1900)

simple polytope:

- "never" first-order rigid. $\leftarrow$ we will focus on these

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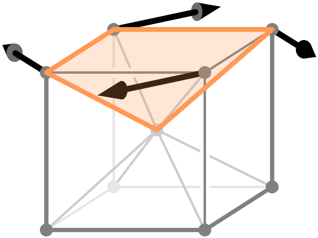

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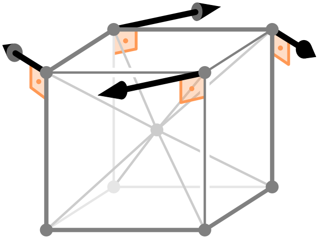

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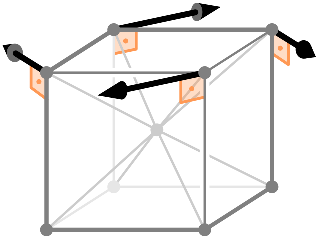

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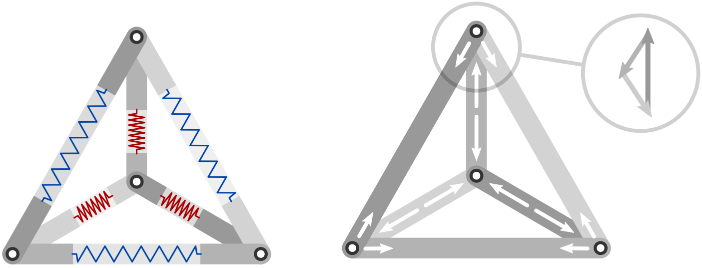
$$\#\text{DOFs} - \#\text{constraints} = d(|V|+1) - (\underbrace{|E|}_{=d/2|V|} + |V|) = \dots = (d/2 - 1)|V| + d \overset{?}{\geq} \#\text{trivial flexes}$$

Stresses

A **stress** of $(G, \boldsymbol{p})$ is a map $\boldsymbol{\omega}\colon E \to \mathbb{R}$ with

$$\text{for all } i \in V \quad \sum_{j:i\sim j} \omega_{ij}(p_j - p_i) = 0.$$



Rigidity matrix

$$\mathcal{R} = \mathcal{R}(G, \boldsymbol{p}) = \begin{array}{c} \overbrace{\begin{array}{ccc} i \in V & \quad & j \in V \end{array}}^{d \times \#\text{vertices}} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \begin{array}{l} \\ \cdots \; ij \in E \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \#\text{edges}$$

$$\ker(\mathcal{R}) = \{\, \text{first-order flexes } \dot{\boldsymbol{p}} \,\}, \qquad \text{i.e. } \mathcal{R}\dot{\boldsymbol{p}} = 0$$

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$$
\begin{aligned}
\ker(\mathcal{R}) &= \{\, \text{first-order flexes } \dot{\boldsymbol{p}} \,\}, && \text{i.e. } \mathcal{R}\dot{\boldsymbol{p}} = 0 \\
\operatorname{coker}(\mathcal{R}) &= \{\, \text{stresses } \boldsymbol{\omega} \,\}. && \text{i.e. } \mathcal{R}^{\top}\boldsymbol{\omega} = 0
\end{aligned}
$$

Stresses and flexes don't usually coexist

If there are <u>many</u> edges ...

- we find many stresses, but no first-order flexes.

If there are <u>few</u> edges ...

- we find many first-order flexes, but no stresses.

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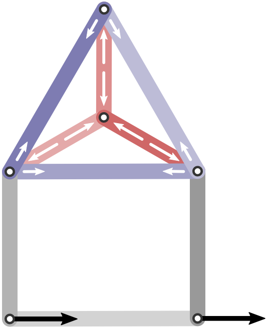

Hands-on: CPFs

Simplicial polytopes:

- potentially many stresses.

Simple polytopes:

- exactly one stress.

Hands-on: CPFs

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Simple polytopes:

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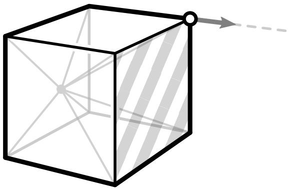
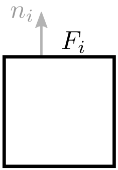
The stress emerges <u>only</u> if all faces are flat.



Paradigm: "stressability = flatability" (e.g. Maxwell-Cremona correspondence)

Minkowski's balancing condition



$$0 = \sum_i \mathrm{vol}(F_i) n_i$$

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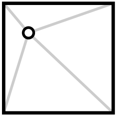

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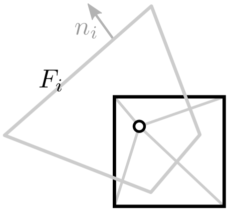

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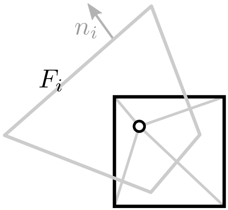

$$0 = \sum_i \mathrm{vol}(F_i) n_i = \sum_i \mathrm{vol}(F_i) \frac{p_i}{\|p_i\|} = \sum_i \overbrace{\frac{\mathrm{vol}(F_i)}{\|p_i\|}}^{\omega_i} p_i = \sum_i \omega_i p_i$$

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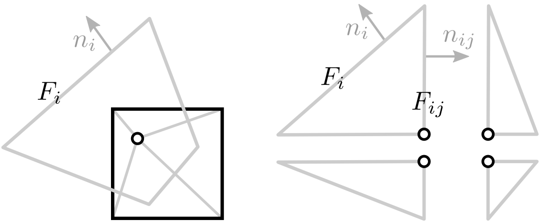

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$$0 = \mathrm{vol}(F_i) n_i + \sum_{j:j\sim i} \mathrm{vol}(F_{ij}) n_{ij} = -\underbrace{\frac{\mathrm{vol}(F_i)}{\|p_i\|}}_{\omega_i}(-p_i) + \sum_{j:j\sim i} \underbrace{\frac{\mathrm{vol}(F_{ij})}{\|p_j - p_i\|}}_{\omega_{ij}}(p_j - p_i)$$

$$= -\omega_i(-p_i) + \sum_{j:j\sim i} \omega_{ij}(p_j - p_i).$$

The Wachspress-Izmestiev stress

The **Wachspress-Izmestiev stress** exists for every CPF:

$\omega_{i\star} =$ **Wachspress coordinate** of the cone point at i-th vertex
$\omega_{ij} = ij$-entry of **Izmestiev matrix**

$$\omega_{i\star} = \frac{\mathrm{vol}(F_i^\diamond)}{\|p_i\|}, \qquad \omega_{ij} = \frac{\mathrm{vol}(F_{ij}^\diamond)}{\|p_i\|\|p_j\| \sin \sphericalangle(p_i, p_j)}.$$

For *simple* CPFs it is the <u>only</u> stress.

Second-order theory

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second-order flex "$=$" does not change edge lengths up to order two.

$$\|p_j^t - p_i^t\| = \|p_j - p_i\| + o(t^2).$$

- next best thing if first-order rigidity fails.
- can provide quantitative bounds on "deformability"

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Theorem. (Connelly, Whiteley, 1996)

- $(G, \boldsymbol{p})$ *is* **second-order rigid** *if "every first-order flex* $\dot{\boldsymbol{p}}$ *is blocked by some stress* $\boldsymbol{\omega}$*", that is*
$$\sum_{ij \in E} \omega_{ij} \|\dot{p}_j - \dot{p}_i\|^2 \neq 0.$$
- $(G, \boldsymbol{p})$ *is* **prestress stable** *if there is a stress* $\boldsymbol{\omega}$ *that "blocks all fo-flexes".*

Hands-on: CPFs

Conjecture.

CPFs are second-order rigid. Moreover, every first-order flex is blocked by the Wachspress-Izmestiev stress.

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Summary

- CPFs are always rigid.
- CPFs are *not* always first-order rigid (never if they are simple).
- We don't know whether CPFs are second-order rigid.

✗		?		✓
first-order rigid	$\rightarrow$	second-order rigid	$\rightarrow$	rigid

The Stress-Flex Conjecture

A mysterious observation

The stress-flex conjecture (Connelly, Gortler, Theran, W.)

Let $(G_P^\star, \boldsymbol{p})$ *be a coned polytope framework. For any choice of*

- *first-order flex* $\dot{\boldsymbol{p}}\colon V \to \mathbb{R}^d$, *with* $\dot{p}_\star = 0$
- *stress* $\boldsymbol{\omega}\colon E \to \mathbb{R}$, *we write* $\omega_i := \omega_{i\star}$

holds

$$\sum_{i \neq \star} \omega_i \dot{p}_i = 0. \quad \longleftarrow \text{ stress-flex orthogonality}$$

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"Corollary"

CPFs are second-order rigid. (actually, prestress stable)

Extent of the conjecture

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  Not true for rigidity or second-order rigidity!

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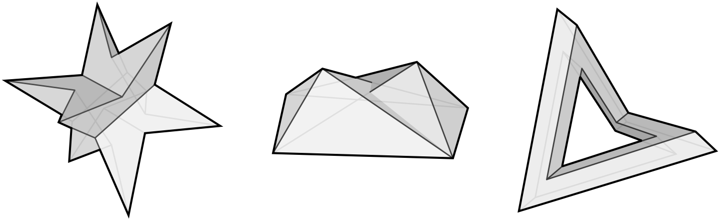

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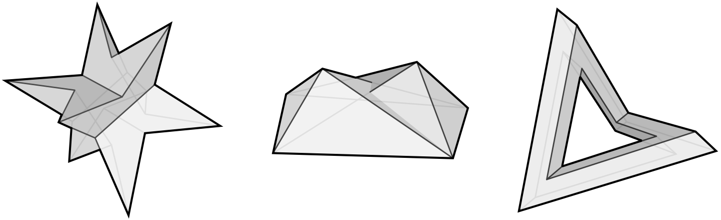
Conclusion: might be less about polytopes and more about *closed PL-surfaces*.

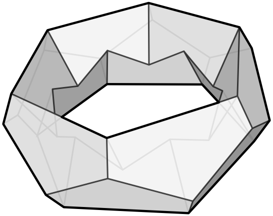


Extended conjecture

Conjecture.

Let $S \subset \mathbb{R}^d$ be a closed PL-surface and $x \in \mathbb{R}^d$ some point. Let $(G_S^\star, \boldsymbol{p})$ be the coned 1-skeleton. If $\dot{\boldsymbol{p}}$ is a first-order flex and $\boldsymbol{\omega}$ is a stress, then

$$\sum_{i \neq \star} \omega_i \dot{p}_i = 0.$$



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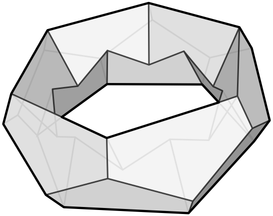
Question: Does stress-flex orthogonality ever not hold?



STRESS-FLEX ORTHOGONALITY HOLDS GENERICALLY

Observation (DEWAR)

Let $(G^\star, \boldsymbol{p})$ be a generic coned framework. Let $\dot{\boldsymbol{p}}$ be a first-order flex and $\boldsymbol{\omega}$ a stress. Then

$$\sum_{i \neq \star} \omega_i \dot{p}_i = 0.$$

Stress-flex orthogonality holds generically

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Let $(G^\star, \boldsymbol{p})$ be a generic coned framework. Let $\dot{\boldsymbol{p}}$ be a first-order flex and $\boldsymbol{\omega}$ a stress. Then

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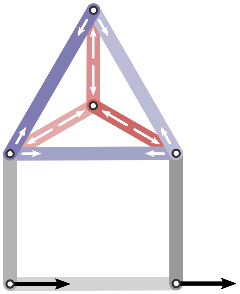
Intuition:

- stresses and flexes live on different parts of a framework.



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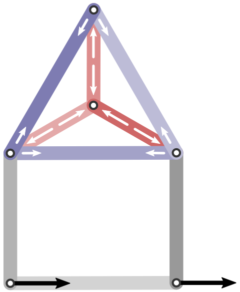
Intuition:

- stresses and flexes live on different parts of a framework.

But ... CPFs are very non-generic

Better question:

- Why does stress-flex orthogonality still hold?



“CHOICE OF STRESS” MIGHT BE A RED HERRING

The stress-flex conjecture asks us to choose

- any first-order flex $\dot{\boldsymbol{p}}$, and
- any stress $\boldsymbol{\omega}$.

But ... this freedom of choice might be a *red herring*!

Maybe ...

- ... only the Wachspress-Izmestiev stress is relevant?
- ... all stresses are generic *except* for the Wachspress-Izmestiev stress?
- ... solving the stress-flex conjecture will teach us something about Wachspress Geometry.

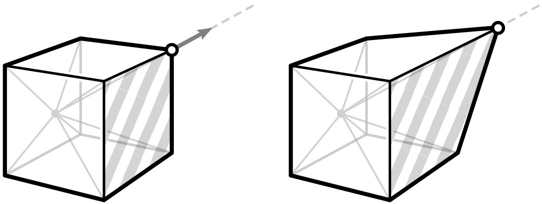
Is it really about closed surfaces?

Question: Does stress-flex orthogonality ever not hold?

generic coned frameworks $\subset$ coned framework with overlapping stresses/flexes $\subset$ coned surfaces.

What else has coexisting stresses and flexes?

Non-example I



Lemma.

First-order flexes and stresses of coned frameworks are preserved by moving vertices radially.

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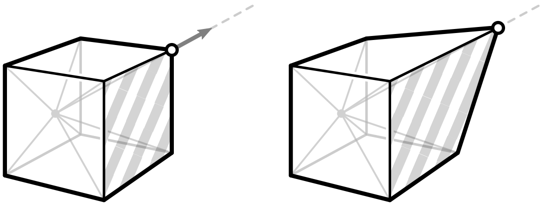


Lemma.

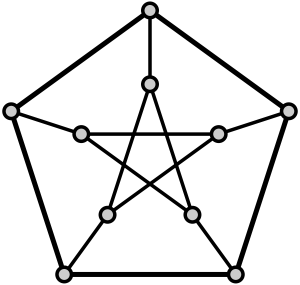
First-order flexes and stresses of coned frameworks are preserved by moving vertices radially.

Observation: Moving vertices radially destroys flex-stress orthogonality.

Non-example II

Spectral embeddings of sparse graphs have stresses and flexes!

... in fact, CPFs are spectral embeddings (Izmestiev, 2007)



Non-example II

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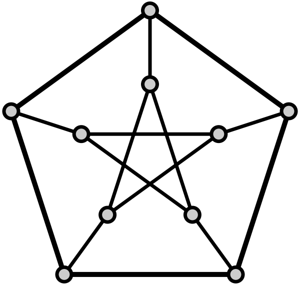

Observation: General spectral embeddings do *not* satisfy stress-flex orthogonality.

... e.g. 4- and 5-dimensional embeddings of Petersen graph.

Reformulation & generalization

Minkowski's
balancing condition

$$0 = \sum_i V_i n_i \quad\Longrightarrow\quad 0 = \frac{\partial}{\partial t}\sum_i V_i n_i = \sum_i \dot V_i n_i + \sum_i V_i \dot n_i.$$

Conjecture.

If there is no first-order change in the angles between adjacent facets, then

$$\sum_i \dot V_i n_i = \sum_i V_i \dot n_i = 0.$$

Thank you.

- M. Winter, *"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"* (2023)
- R. Connelly, S. J. Gortler, L. Theran, M. Winter, *"Energies on Coned Convex Polytopes"* (2024)
- R. Connelly, S. J. Gortler, L. Theran, M. Winter, *"The Stress-Flex Conjecture"* (2024)

- September 24th – 25th, University of Leipzig
  Workshop "Wachspress Geometry"

A possible approach

$$\sum_{i \neq \star} \omega_i p_i = \sum_i \frac{\mathrm{vol}(F_i^\diamond)}{\|p_i\|} = \sum_i \mathrm{vol}(F_i^\diamond) \frac{p_i}{\|p_i\|} = \sum_i \underbrace{\mathrm{vol}(F_i^\diamond)}_{=:V_i} n_i = \sum V_i n_i = 0.$$

For $(G, \boldsymbol{p}^t)$ define

$$P^{t\circ} := \{x \in \mathbb{R}^d \mid \langle p_i^t, x \rangle \leq 1 \text{ for all } i \neq \star\}.$$

This gives t-dependent V_i^t and n_i^t, but we suppress the t-s.

$$0 = \sum_i V_i n_i \quad \Longrightarrow \quad 0 = \frac{\partial}{\partial t} \sum_i V_i n_i = \sum_i \dot{V}_i n_i + \sum_i V_i \dot{n}_i.$$

Conjecture.

If there is no first-order change in facet-origin distance and the angles between adjacent facets, then

$$\sum_i \dot{V}_i n_i = \sum_i V_i \dot{n}_i = 0.$$

The dual picture

Infinitesimal displacement of the vertices so that

- vertex-origin distances stay the same. (in first order)
- polytope edge lengths stay the same. (in first order)

show $\sum_i \omega_i \dot{p}_i = \sum_i \dot{\omega}_i p_i = 0$.

$$\Updownarrow \quad \text{polar duality}$$

Infinitesimal displacement of the facet hyperplanes so that

- facet-origin distances stay the same. (in first order)
- dihedral angles stay the same. (in first order)

show $\sum_i V_i \dot{n}_i = \sum_i \dot{V}_i n_i = 0$.