Rigidity of Polyhedral Spheres beyond Triangulations

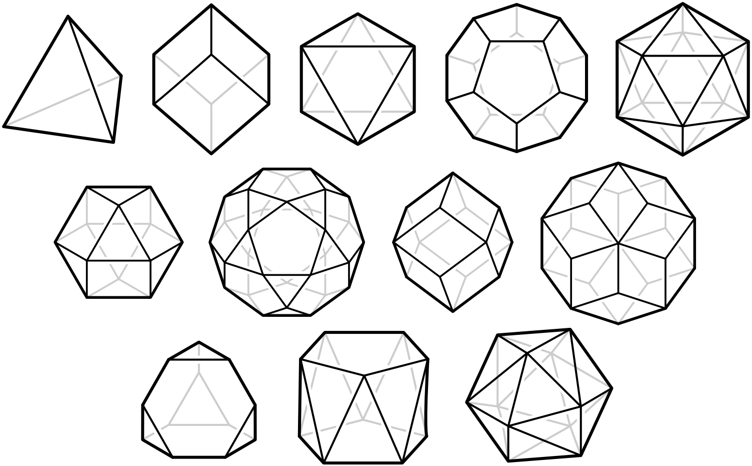
Martin Winter

University of Warwick

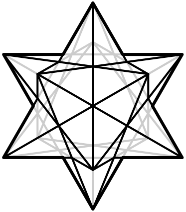
March 6, 2024

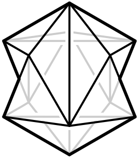
Joint work with Bernd Schulze, Matthias Himmelmann, Albert Zhang

Triangular and polyhedral spheres

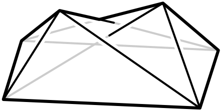


Non-convex and self-intersecting





Jessen's icosahedron



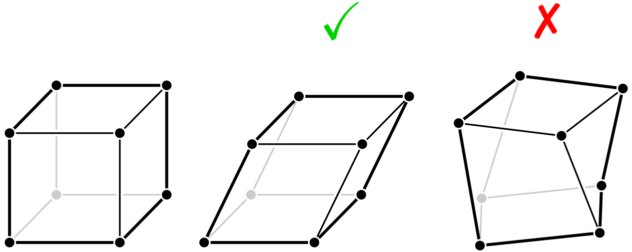
Bricard octahedron

Polyhedral spheres

"A polyhedral sphere is a bunch of polygons glued edge to edge so that they form a topological sphere."

- a **polyhedral sphere** $\mathcal{P} = (V, E)$ is a $\overbrace{\text{3-connected planar graph}}^{\text{polyhedral graph}}$.
- its faces we denote by $F_1, ..., F_m \subset V$.
- a **realization** of $\mathcal{P}$ is a map $\boldsymbol{p} : V \to \mathbb{R}^3$ so that the points $p_i, i \in F_k$ lie on a common plane.
- in a **triangulated sphere** all faces are triangles.

Flexing polyhedral spheres

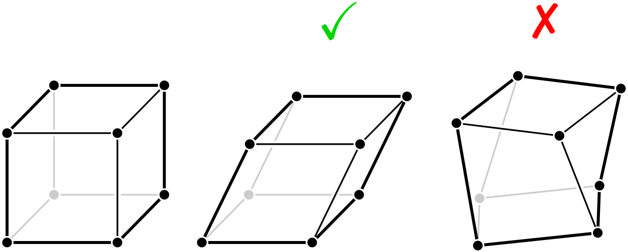


- preserving edge lengths

  but also

- preserve planarity of faces

Flexing polyhedral spheres



- preserving edge lengths

  but also

- preserve planarity of faces

$$\#\text{DOFs} - \#\text{constraints} = 3|V| - \Big(|E| + \sum_k (|F_k| - 3)\Big) = 6.$$

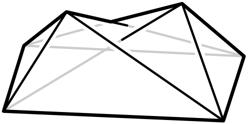
Triangulated Spheres

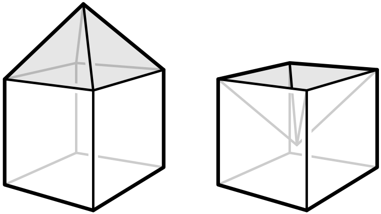
... good old frameworks

Rigidity of triangulated spheres

Core results

- *Convex* triangulated spheres are *globally* rigid. (Cauchy)
- *Convex* triangulated spheres are *first-order* rigid. (Dehn)
- Triangulated spheres are *generically first-order* rigid. (Gluck)
- *Flexible* triangulated spheres exist. (Bricard, Connelly, Steffen)



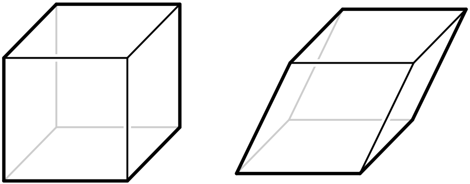


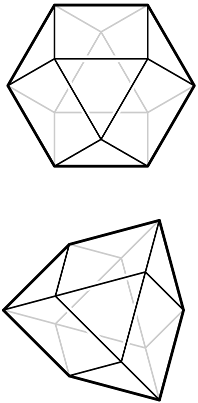
Moving beyond triangulations

Rigidity of general polyhedral spheres

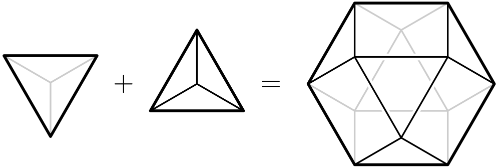
Core results

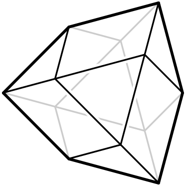
- *Convex* polyhedra with *fixed face shapes* are *globally* rigid. (Cauchy)
  (also in higher dimensions) (Alexandrov)
- *Triangulating* a convex polyhedron makes it *first-order* rigid. (Alexandrov)



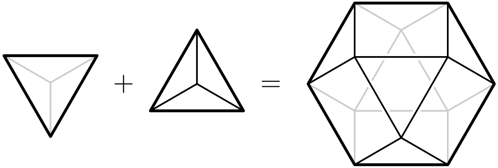


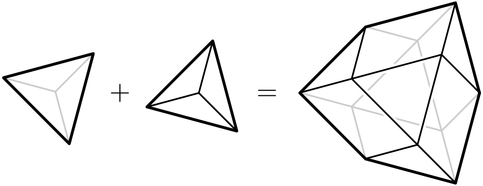
Minkowski sums $A + B := \{a + b \mid a \in A, b \in B\}$





Minkowski sums $A + B := \{a + b \mid a \in A, b \in B\}$



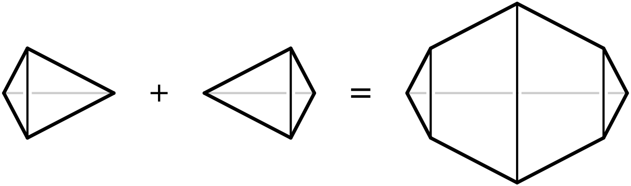


Only Minkowski sums?

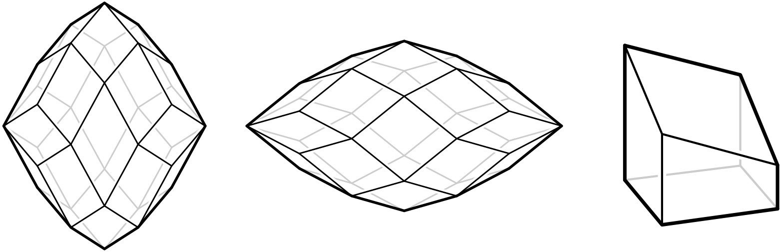
Question: Are all flexible convex polyhedra Minkowski sums?

Notes:

- This includes rotating/flexing a proper Minkowski summand.
- Not all Minkowski sums are flexible.

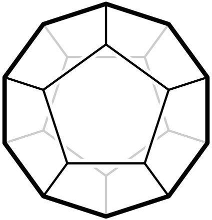


AFFINE FLEXES := A FLEX REALIZED BY AN AFFINE TRANSFORMATION

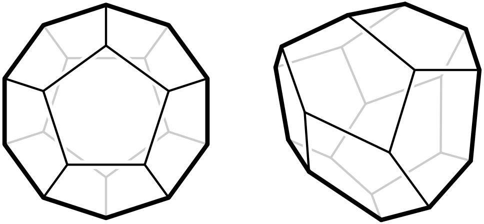


Question: Are all affinely flexible polyhedra Minkowski sums?

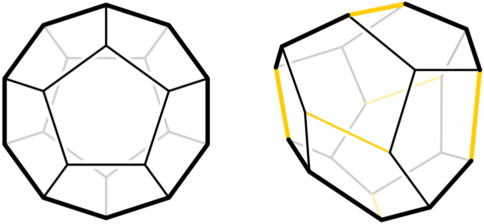
Is the regular dodecahedron rigid?



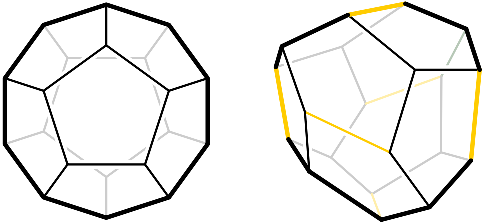
Is the regular dodecahedron rigid?



Is the regular dodecahedron rigid?



Is the regular dodecahedron rigid?

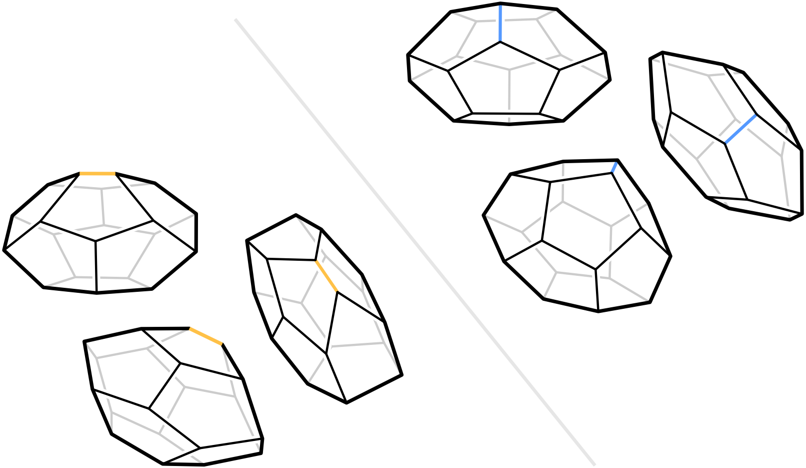


Theorem. (Himmelmann, Schulze, W., Zhang, 2024+)

The regular dodecahedron is ...

- ✗ *not first-order rigid.* (5-dimensional space of first-order flexes)
- ✗ *not prestress stable.*
- ✓ *second-order rigid.*

No generic global rigidity



Many open questions

Question: (about convex spheres)

- Is second order rigidity always sufficient?
- Does flexibility need parallel edges?
- Is polytope rigidity preserved under affine transformations?

  (first-order flexibility is not)

Generic first-order rigidity

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \right\}$$

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{matrix} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{matrix} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\; \text{if } i \in F_k \right\}$$

$$\textsc{Real-cvx}(\mathcal{P}) := \left\{ \begin{matrix} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{matrix} \;\middle|\; \begin{matrix} \langle p_i, n_k \rangle = 1 \;\; \text{if } i \in F_k \\ \langle p_i, n_k \rangle < 1 \;\; \text{if } i \notin F_k \end{matrix} \right\}$$

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k\rangle = 1 \;\; \text{if } i \in F_k \right\}$$

$$\textsc{Real-cvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \begin{array}{ll} \langle p_i, n_k\rangle = 1 & \text{if } i \in F_k \\ \langle p_i, n_k\rangle < 1 & \text{if } i \notin F_k \end{array} \right\}$$

A **finite flex** preserves

$$\|p_i - p_j\| \overset{!}{=} \ell_{ij} = \text{const} \qquad \text{for } ij \in E$$

$$\langle p_i, n_k\rangle \overset{!}{=} 1 \qquad \text{for } i \in F_k$$

Main result

Theorem. (Himmelmann, Schulze, W., Zhang, 2024)

Convex polyhedral spheres are generically first-order rigid.

$$\mathrm{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \,\middle|\, \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \right\}$$

$$\mathrm{Real\text{-}cvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \,\middle|\, \begin{array}{ll} \langle p_i, n_k \rangle = 1 & \text{if } i \in F_k \\ \langle p_i, n_k \rangle < 1 & \text{if } i \notin F_k \end{array} \right\}$$

A **finite flex** preserves

$$\begin{aligned} \|p_i - p_j\| &\overset{!}{=} \ell_{ij} = \text{const} && \text{for } ij \in E \\ \langle p_i, n_k \rangle &\overset{!}{=} 1 && \text{for } i \in F_k \end{aligned}$$

A **first-order flex** $(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})$ satisfies

$$\begin{aligned} \langle p_i - p_j, \dot p_i - \dot p_j \rangle &= 0 && \text{for } ij \in E \\ \langle p_i, \dot n_k \rangle + \langle \dot p_i, n_k \rangle &= 0 && \text{for } i \in F_k \end{aligned}$$

The Proof

– the triangular case –

"Triangular spheres are generically first-order rigid."

Rigidity matrix $\mathcal{R}(G, \boldsymbol{p})$

$$\overbrace{\quad i \in V \qquad\qquad j \in V \quad}^{3\ \times\ \#\text{vertices}}$$

$$\#\text{edges} \left\{ ij \in E \right. \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix} = \mathcal{R}(G, \boldsymbol{p})$$

$\dot{\boldsymbol{p}}$ is a first-order flex $\iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

Rigidity matrix $\mathcal{R}(G, \boldsymbol{p})$

$$\#\text{edges}\left\{ ij \in E \quad \overbrace{\begin{pmatrix} & i \in V & & j \in V & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \end{pmatrix}}^{3\,\times\,\#\text{vertices}} \right. = \mathcal{R}(G, \boldsymbol{p})$$

$\dot{\boldsymbol{p}}$ is a first-order flex $\iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

$(G, \boldsymbol{p})$ is first-order rigid $\iff \operatorname{corank} \mathcal{R}(G, \boldsymbol{p}) = 6$

Rigidity matrix $\mathcal{R}(G,\boldsymbol{p})$

$$\#\text{edges}\left\{\; ij\in E \;\; \overbrace{\begin{pmatrix} & i\in V & & j\in V & \\ & \vdots & & \vdots & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix}}^{3\,\times\,\#\text{vertices}} \right. = \mathcal{R}(G,\boldsymbol{p})$$

$\dot{\boldsymbol{p}}$ is a first-order flex $\iff$ $\mathcal{R}(G,\boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$

$(G,\boldsymbol{p})$ is first-order rigid $\iff$ $\operatorname{corank}\mathcal{R}(G,\boldsymbol{p}) = 6$

#columns − #rows $= 3|V| - |E| = 6 =$ #trivial first-order flexes.

Rigidity matrix $\mathcal{R}(G, \boldsymbol{p})$

$$\#\text{edges} \left\{ \; ij \in E \; \begin{pmatrix} & \overbrace{i \in V \qquad\qquad j \in V}^{3 \,\times\, \#\text{vertices}} & \\ \cdots & p_i - p_j \quad \cdots \quad p_j - p_i & \cdots \end{pmatrix} \right. = \mathcal{R}(G, \boldsymbol{p})$$

$$\dot{\boldsymbol{p}} \text{ is a first-order flex } \iff \mathcal{R}(G, \boldsymbol{p})\,\dot{\boldsymbol{p}} = 0$$

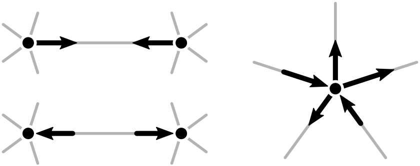
$$(G, \boldsymbol{p}) \text{ is first-order rigid } \iff \operatorname{corank} \mathcal{R}(G, \boldsymbol{p})^\top = 0$$

$$\#\text{columns} - \#\text{rows} \;=\; 3|V| - |E| \;=\; 6 \;=\; \#\text{trivial first-order flexes.}$$

Stresses

$$\mathcal{R}(G,\boldsymbol{p})^\top \boldsymbol{\omega} = 0.$$

$$\forall i \in V\colon \sum_{j:ij\in E} \omega_{ij}(p_j - p_i) = 0.$$



first-order flexible $\iff$ $\ker \mathcal{R}(G,\boldsymbol{p})^\top = \{0\}$ $\iff$ $\exists$ non-zero stress

Generic first-order rigidity

$\textsc{Real}(G) := \{\, \text{3-dimensional frameworks on } G \,\} = \mathbb{R}^{3V}$

$$
\begin{aligned}
\textsc{Flex}(G) &:= \{\, \text{first-order flexible frameworks on } G \,\} \\
&= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \operatorname{rank} \mathcal{R}(G, \boldsymbol{p}) < |E| \,\} \\
&= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \det(A) = 0 \text{ for all } |E| \times |E| \text{ submatrices } A \text{ of } \mathcal{R}(G, \boldsymbol{p}) \,\}.
\end{aligned}
$$

Generic first-order rigidity

$\textsc{Real}(G) := \{\,\text{3-dimensional frameworks on } G\,\} = \mathbb{R}^{3V}$

$$\begin{aligned}\textsc{Flex}(G) &:= \{\,\text{first-order flexible frameworks on } G\,\}\\ &= \{\,\boldsymbol{p}\in\mathbb{R}^{3V} \mid \operatorname{rank}\mathcal{R}(G,\boldsymbol{p}) < |E|\,\}\\ &= \{\,\boldsymbol{p}\in\mathbb{R}^{3V} \mid \det(A) = 0 \text{ for all } |E|\times|E| \text{ submatrices } A \text{ of } \mathcal{R}(G,\boldsymbol{p})\,\}.\end{aligned}$$

$\Longrightarrow$ $\textsc{Flex}(G) \subseteq \mathbb{R}^{3V}$ is the zero set of polynomials.

$\Longrightarrow$ either $\textsc{Flex}(G) = \mathbb{R}^{3V}$ or $\textsc{Flex}(G)$ has measure zero.

Generic first-order rigidity

$\textsc{Real}(G) := \{\,$3-dimensional frameworks on $G\,\} = \mathbb{R}^{3V}$

$\textsc{Flex}(G) := \{\,$first-order flexible frameworks on $G\,\}$

$$= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \operatorname{rank} \mathcal{R}(G, \boldsymbol{p}) < |E| \,\}$$

$$= \{\, \boldsymbol{p} \in \mathbb{R}^{3V} \mid \det(A) = 0 \text{ for all } |E| \times |E| \text{ submatrices } A \text{ of } \mathcal{R}(G, \boldsymbol{p}) \,\}.$$

$\implies$ $\textsc{Flex}(G) \subseteq \mathbb{R}^{3V}$ is the zero set of polynomials.

$\implies$ either $\textsc{Flex}(G) = \mathbb{R}^{3V}$ or $\textsc{Flex}(G)$ has measure zero.

Recall: a convex realization is first-order rigid. (Dehn)

$\implies$ $\textsc{Flex}(G) \neq \mathbb{R}^{3V}$.

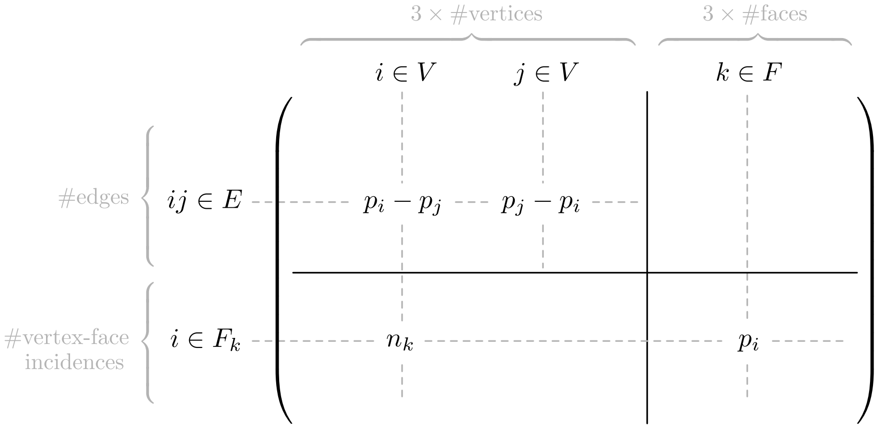
$\implies$ $\textsc{Flex}(G)$ has measure zero.

The Proof

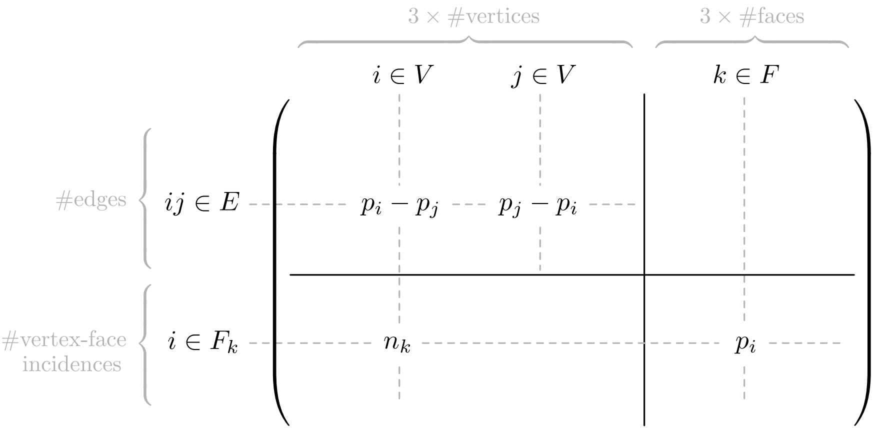
– the polyhedral case –

"Convex polyhedral spheres are generically first-order rigid."

Rigidity matrix $\mathcal{R}(P)$



Rigidity matrix $\mathcal{R}(P)$



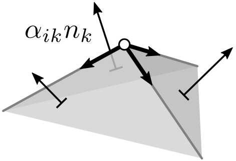
$$\#\text{columns} - \#\text{rows} = \big(3|V| + 3|F|\big) - \big(|E| + |VF|\big) = 6$$

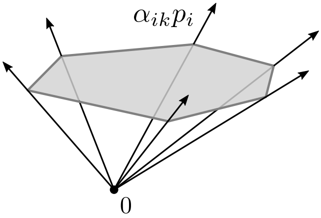
Stresses

$$\mathcal{R}(P)^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$$

$$\forall i \in V: \quad 0 = \sum_{j:ij \in E} \omega_{ij}(p_j - p_i) + \sum_{k:i \in F_k} \alpha_{ik} n_k$$

$$\forall k \in F: \quad 0 = \sum_{i \in F_k} \alpha_{ik} p_i$$



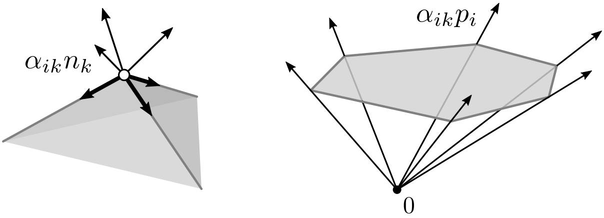


Stresses

$$\mathcal{R}(P)^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$$

$$\forall i \in V: \quad 0 = \sum_{j:ij\in E} \omega_{ij}(p_j - p_i) + \sum_{k:i\in F_k} \alpha_{ik} n_k$$

$$\forall k \in F: \quad 0 = \sum_{i\in F_k} \alpha_{ik} p_i$$

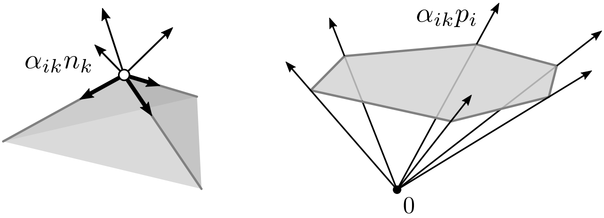


Stresses

$$\mathcal{R}(P)^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$$

$$\forall i \in V: \quad 0 = \sum_{j: ij \in E} \omega_{ij}(p_j - p_i) + \sum_{k: i \in F_k} \alpha_{ik} n_k$$

$$\forall k \in F: \quad 0 = \sum_{i \in F_k} \alpha_{ik} p_i$$



Observation: If F_k is *triangular* face, then $\alpha_{ik} = 0$.

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall \mathcal{P} \quad \textsc{Flex}(\mathcal{P}) \text{ has measure zero in } \textsc{Real-cvx}(\mathcal{P}).$$

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall\, \mathcal{P} \quad \textsc{Flex}(\mathcal{P}) \text{ has measure zero in } \textsc{Real-cvx}(\mathcal{P}).$$

Question: What about potentially non-convex polyhedral spheres?

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall \mathcal{P} \quad \text{Flex}(\mathcal{P}) \text{ has measure zero in } \text{Real-cvx}(\mathcal{P}).$$

Question: What about potentially non-convex polyhedral spheres?

Strategy:

1. Polynomial method: $\text{Flex}(\mathcal{P}) \subseteq \text{Real}(\mathcal{P})$ is a sub-variety.
   $\longrightarrow$ $\text{Flex}(\mathcal{P}) = \text{Real}(\mathcal{P})$ *or*
   $\text{Flex}(\mathcal{P})$ has measure zero in $\text{Real}(\mathcal{P})$.
2. Show: there exists at least one realization that is first-order rigid.

Generic rigidity of polyhedral spheres

Theorem. (Himmelmann, Schulze, W., Zhang)

Convex polyhedral spheres are generically first-order rigid.

$$\forall \mathcal{P} \quad \textsc{Flex}(\mathcal{P}) \textit{ has measure zero in } \textsc{Real-cvx}(\mathcal{P}).$$

Question: What about potentially non-convex polyhedral spheres?

Strategy:

1. Polynomial method: $\textsc{Flex}(\mathcal{P}) \subseteq \textsc{Real-cvx}(\mathcal{P})$ is a sub-variety.
   $\longrightarrow$ $\textsc{Flex}(\mathcal{P}) = \textsc{Real-cvx}(\mathcal{P})$ *or*
   $\textsc{Flex}(\mathcal{P})$ has measure zero in $\textsc{Real-cvx}(\mathcal{P})$.
2. Show: there exists at least one convex realization that is first-order rigid.

Theorem. (Steinitz)

$\textsc{Real-cvx}(\mathcal{P}) \subset \mathbb{R}^{3V} \times \mathbb{R}^{3F}$ *is (an open subset of) a smooth, irreducible, contractible variety of dimension* $|E| + 6$.

Reduction to existence

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\; \text{if } i \in F_k \right\}$$

$$\textsc{Real-cvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \begin{array}{ll} \langle p_i, n_k \rangle = 1 & \text{if } i \in F_k \\ \langle p_i, n_k \rangle < 1 & \text{if } i \notin F_k \end{array} \right\}$$

$$\begin{aligned} \textsc{Flex}(\mathcal{P}) &:= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid (\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) \text{ is first-order flexible} \} \\ &= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid \operatorname{rank} \mathcal{R}(\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) < |E| + |VF| \} \end{aligned}$$

$\implies$ $\textsc{Flex}(G) \subseteq \textsc{Real-cvx}(P)$ is the zero set of polynomials.

$\implies$ either $\textsc{Flex}(\mathcal{P}) = \textsc{Real-cvx}(P)$
or $\textsc{Flex}(\mathcal{P})$ has measure zero in $\textsc{Real-cvx}(P)$.

Reduction to existence

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \middle| \; \langle p_i, n_k \rangle = 1 \;\; \text{if } i \in F_k \right\}$$

$$\textsc{Real-cvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \middle| \; \begin{array}{ll} \langle p_i, n_k \rangle = 1 & \text{if } i \in F_k \\ \langle p_i, n_k \rangle < 1 & \text{if } i \notin F_k \end{array} \right\}$$

$$\begin{aligned} \textsc{Flex}(\mathcal{P}) &:= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid (\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) \text{ is first-order flexible} \} \\ &= \{ (\boldsymbol{p}, \boldsymbol{n}) \in \textsc{Real-cvx}(P) \mid \operatorname{rank} \mathcal{R}(\mathcal{P}, \boldsymbol{p}, \boldsymbol{n}) < |E| + |VF| \} \end{aligned}$$

$\Longrightarrow$ $\textsc{Flex}(G) \subseteq \textsc{Real-cvx}(P)$ is the zero set of polynomials.

$\Longrightarrow$ either $\textsc{Flex}(\mathcal{P}) = \textsc{Real-cvx}(P)$
or $\textsc{Flex}(\mathcal{P})$ has measure zero in $\textsc{Real-cvx}(P)$.

It remains to show: there exists a first-order rigid convex realization.

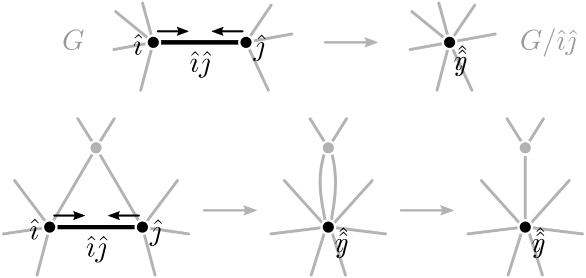
THE PROOF

– proving existence –

"There exists at least one first-order rigid realization."

Strategy: INDUCTION ON #EDGES

Decreasing the edge number by **contraction**:



Theorem. (TUTTE)

If $G \neq K_4$ is 3-connected, there is an edge $e \in E$ for which G/e is 3-connected.

Strategy: INDUCTION ON #EDGES

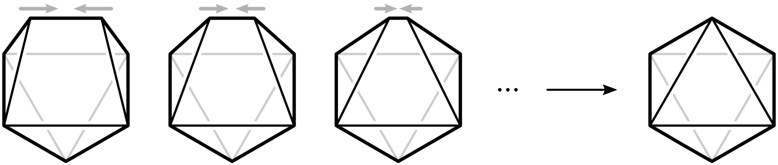
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- Show: if each P_i has a non-zero stress, so does P'. ↯

  $\longrightarrow$ some P_i must be first-order rigid.



Strategy: INDUCTION ON #EDGES

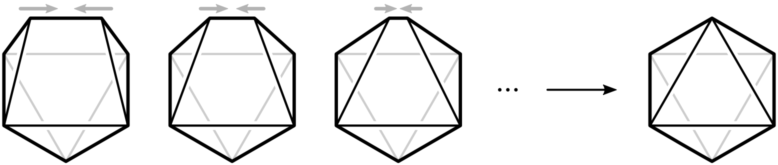
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- ? Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- ? Show: if each P_i has a non-zero stress, so does P'. ↯

  $\longrightarrow$ some P_i must be first-order rigid.



Stresses survive contraction

Given a sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$ realizing the contracting $\hat\imath\hat\jmath \longrightarrow \hat{\hat{\jmath}}$.

Lemma.

If each P_n has a non-zero stress $(\boldsymbol{\omega}^n, \boldsymbol{\alpha}^n)$, then P' also has a non-zero stress $(\boldsymbol{\omega}', \boldsymbol{\alpha}')$.

$$
\begin{aligned}
\omega_{ij}^n &\longrightarrow \omega'_{ij} && \text{if } i,j \notin \{\hat\imath,\hat\jmath\} \\
\omega_{i\hat\imath}^n + \omega_{i\hat\jmath}^n &\longrightarrow \omega'_{i\hat{\hat{\jmath}}} && \text{if } i \notin \{\hat\imath,\hat\jmath\} \\
\omega_{\hat\imath\hat\jmath}^n &\longrightarrow - \\
\alpha_{ik}^n &\longrightarrow \alpha'_{ik} && \text{if } i \notin \{\hat\imath,\hat\jmath\} \\
\alpha_{\hat\imath k}^n + \alpha_{\hat\jmath k}^n &\longrightarrow \alpha'_{\hat{\hat{\jmath}}k}
\end{aligned}
$$

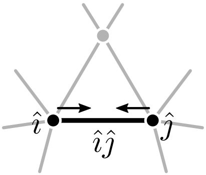
Stresses survive contraction

Given a sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$ realizing the contracting $\hat\imath\hat\jmath \longrightarrow \hat{\hat{\jmath}}$.

Lemma.

If each P_n has a non-zero stress $(\boldsymbol{\omega}^n, \boldsymbol{\alpha}^n)$, then P' also has a non-zero stress $(\boldsymbol{\omega}', \boldsymbol{\alpha}')$.

$$
\begin{aligned}
\omega_{ij}^n &\longrightarrow \omega'_{ij} && \text{if } i,j \notin \{\hat\imath,\hat\jmath\} \\
\omega_{i\hat\imath}^n + \omega_{i\hat\jmath}^n &\longrightarrow \omega'_{i\hat{\hat{\jmath}}} && \text{if } i \notin \{\hat\imath,\hat\jmath\} \\
\omega_{\hat\imath\hat\jmath}^n &\longrightarrow - && \\
\alpha_{ik}^n &\longrightarrow \alpha'_{ik} && \text{if } i \notin \{\hat\imath,\hat\jmath\} \\
\alpha_{\hat\imath k}^n + \alpha_{\hat\jmath k}^n &\longrightarrow \alpha'_{\hat{\hat{\jmath}}k} &&
\end{aligned}
$$



Note: if F_k is a triangle, then $\alpha_{ik} = 0$.

Strategy: INDUCTION ON #EDGES

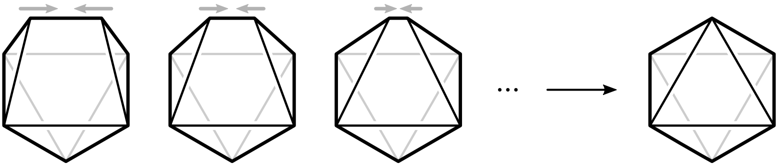
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

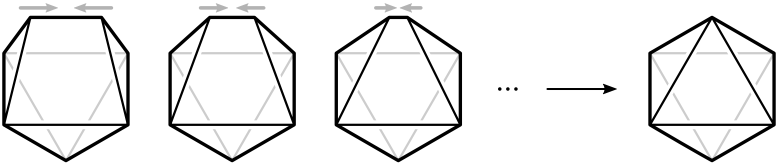
- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- ? Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- ✓ Show: if each P_i has a non-zero stress, so does P'. ↯

  $\longrightarrow$ some P_i must be first-order rigid.



Contracting edges geometrically

How to find the sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$'?



- Maxwell-Cremona correspondence

  "Polyhedral realizations are in 1:1 relation with planar stressed frameworks."

- Tutte embedding

  "One can prescribe the stresses of a planar framework."

Project and lift

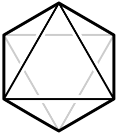
Input: P'

Output: sequence $P_1, P_2, P_3, ... \longrightarrow P'$

Project and lift

Input: P'

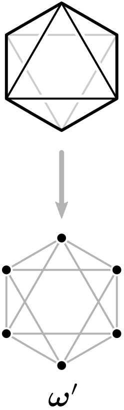
Output: sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$



Project and lift

Input: P'

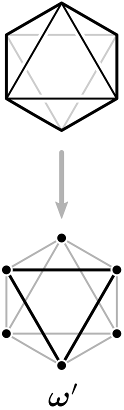
Output: sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$



Project and lift

Input: P'

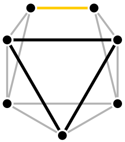
Output: sequence $P_1, P_2, P_3, \dots \longrightarrow P'$



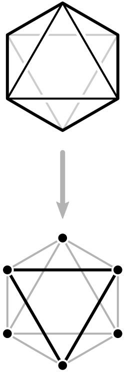
Project and lift

Input: P'

Output: sequence $P_1, P_2, P_3, \dots \longrightarrow P'$



ω^1

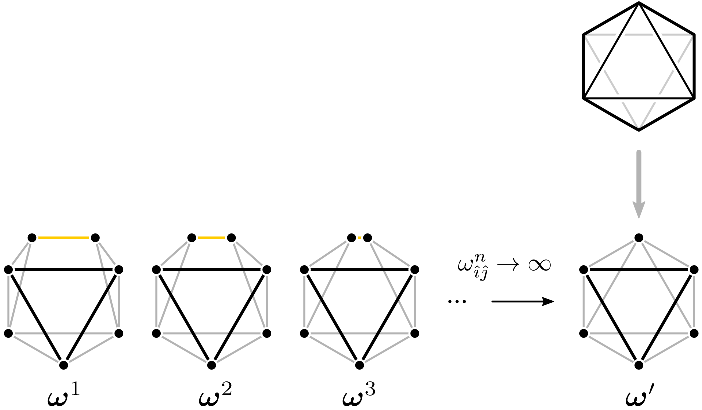


ω'

Project and lift

Input: P'

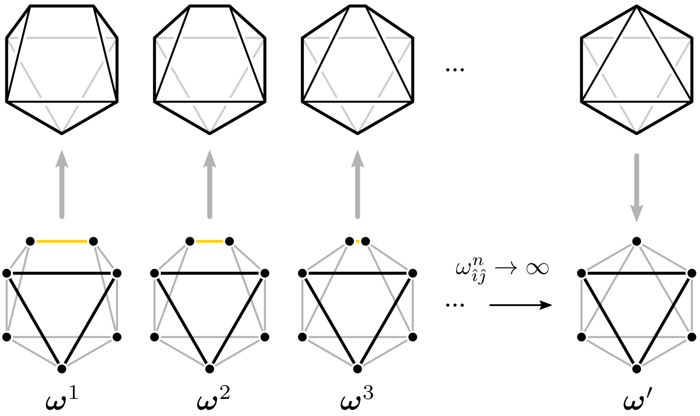
Output: sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$



Project and lift

Input: P'

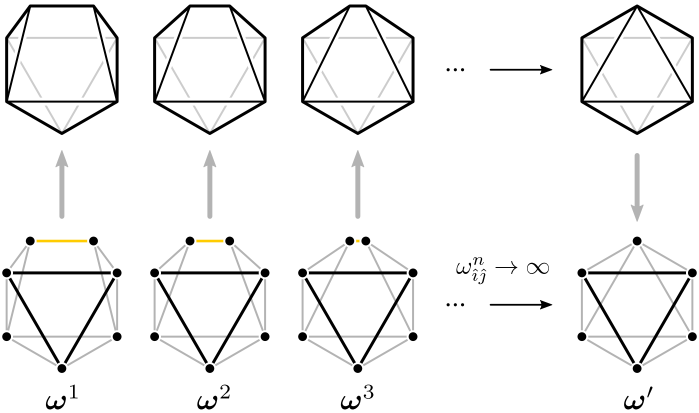
Output: sequence $P_1, P_2, P_3, \ldots \longrightarrow P'$



Project and lift

Input: P'

Output: sequence $P_1, P_2, P_3, \dots \longrightarrow P'$



Strategy: INDUCTION ON #EDGES

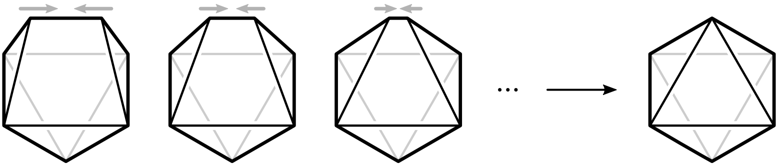
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- ✓ Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- ✓ Show: if each P_i has a non-zero stress, so does P'. ↯

  $\longrightarrow$ some P_i must be first-order rigid.



Strategy: INDUCTION ON #EDGES

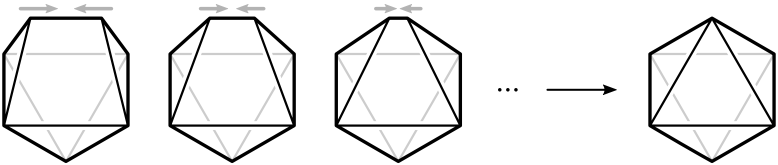
Induction base:

- $|E| = 6$ (simplex) is clearly rigid.

Induction step:

- Choose an edge $e \in E$ for which G_P/e is polyhedral.
- *Induction hypothesis*: there is a first-order rigid realizations P' of G_P/e.
- ✓ Choose a sequence of realizations $P_1, P_2, P_3, \ldots \longrightarrow P'$.
- ✓ Show: if each P_i has a non-zero stress, so does P'. $\lightning$

  $\longrightarrow$ some P_i must be first-order rigid. □



Many open questions

Higher dimensions

Question: Are polytopes of dimension $d \geq 4$ generically first-order rigid?

Problems:

- first-order rigid $\neq$ no non-zero stresses
- realization space is no longer contractible/connected/irreducible/...
- there are no useful analogues of Maxwell-Cremona/Tutte/...

Higher dimensions

Question: Are polytopes of dimension $d \geq 4$ generically first-order rigid?

Problems:

- first-order rigid $\neq$ no non-zero stresses
- realization space is no longer contractible/connected/irreducible/...
- there are no useful analogues of Maxwell-Cremona/Tutte/...

However ... can we pull the result up from dimension three?

- Fix a generic realization $P \subset \mathbb{R}^4$.
  $\longrightarrow$ all facets (= 3-dimensional faces) are generic.
- Suppose P has a first-order flex.
  $\longrightarrow$ induces a first-order flex on each facet.
- Since the facets are generic + 3D, the flexes on each facet must be trivial.
- Show: the flex of P must therefore be trivial as well. (Cauchy, Dehn)

Higher dimensions

Question: Are polytopes of dimension $d \geq 4$ generically first-order rigid?

Problems:

- first-order rigid $\neq$ no non-zero stresses
- realization space is no longer contractible/connected/irreducible/...
- there are no useful analogues of Maxwell-Cremona/Tutte/...

However ... can we pull the result up from dimension three?

- Fix a generic realization $P \subset \mathbb{R}^4$.

? $\longrightarrow$ all facets (= 3-dimensional faces) are generic.

- Suppose P has a first-order flex.

  $\longrightarrow$ induces a first-order flex on each facet.
- Since the facets are generic + 3D, the flexes on each facet must be trivial.

? ▶ Show: the flex of P must therefore be trivial as well. (Cauchy, Dehn)

Beyond convexity

$$\text{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \right\}$$

$$\text{Real-cvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^3 \\ \boldsymbol{n}\colon F \to \mathbb{R}^3 \setminus \{0\} \end{array} \;\middle|\; \begin{array}{l} \langle p_i, n_k \rangle = 1 \;\text{ if } i \in F_k \\ \langle p_i, n_k \rangle < 1 \;\text{ if } i \notin F_k \end{array} \right\}$$

Question: Is $\text{Real}(\mathcal{P})$ irreducible?

or, alternatively,

Question: Can we run the "convex proof" once per irreducible component of $\text{Real}(\mathcal{P})$?

Thank you.