Adjoint degrees and scissors congruence for polytopes

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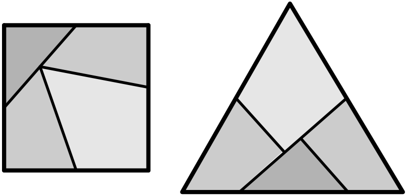
Hilbert's Third Problem

Given any two polyhedra P and Q of equal volume, is it always possible to dissect P into finitely many polyhedral pieces $P_1, ..., P_n$, which can then be reassembled to yield Q?

- $d = 2$: true by the *Wallace–Bolyai–Gerwien theorem*
- $d = 3$: false as shown by Max Dehn using the *Dehn invariant*
  (takes values in $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z})$)
- Marked the beginning of **valuation theory**



Valuations

Whenever P, Q, $P \cap Q$ and $P \cup Q$ are polytopes a **valuation** satisfies

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$$

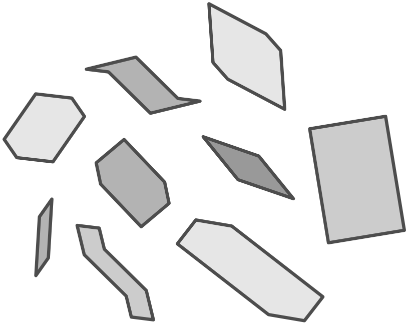
... but what we actually care about:

$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

Examples:

- volume
- surface area measure
- Euler characteristic
- mixed volumes
- number of contained lattice points
- ...

A centrally-symmetric puzzle



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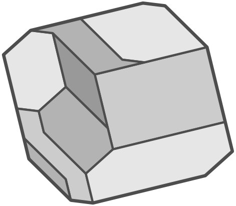

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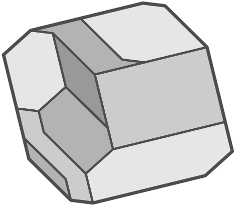
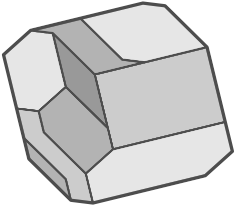


Let $\nu(P)$ be the *surface area measure* of $P \subset \mathbb{R}^d$ on $\mathbb{S}^{d-1}$. Define

$$\phi(P) := \nu(P) - \nu(-P)$$

A CENTRALLY-SYMMETRIC PUZZLE



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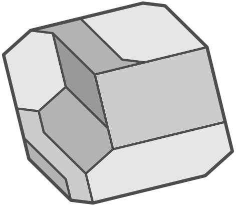

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- $\phi(P_1 \cupdot \cdots \cupdot P_n) = \phi(P_1) + \cdots + \phi(P_n)$ (i.e. ϕ is valuative)
- $\phi(P) = 0$ if and only if P is centrally symmetric.

Everybody's new favourite valuation

Ω

The canonical form

The **canonical form** of a polytope $P \subset \mathbb{R}^d$ is the rational function given by

$$\Omega(P; x) := \mathrm{vol}(P - x)^\circ = \frac{\mathrm{adj}_P(x)}{\prod_F \ell_F(x)}.$$

- the product $\prod_F \ell_F$ is over all facets $F \subset P$.
- $\ell_F(x) := \langle u_F, x \rangle - h_F$ is the facet defining linear form
- u_k is the unit normal vector of F
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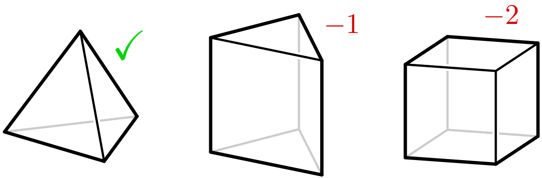
Theorem.

The canonical form is valuative:

$$\Omega(P_1 \cup \cdots \cup P_n; x) = \Omega(P_1;x) + \cdots + \Omega(P_n;x).$$

Adjoint degrees

- Generically (or projectively) the adjoint adj_P has degree $m - d - 1$.
  (where $m = \#$facets)
- This is not true in general.



We call this defficiency in degree the **degree drop** of P:

$$\deg \mathrm{adj}_P = m - d - 1 - \mathrm{drop}(P)$$

Example: for the d-cube $\square^d$ we have

$$\Omega(\square^d; x) = \frac{\text{some constant}}{\prod_i (1 - x_i^2)} \quad \Longrightarrow \quad \mathrm{drop}(\square^d) = d - 1.$$

Adjoint degrees and composition

Lemma.

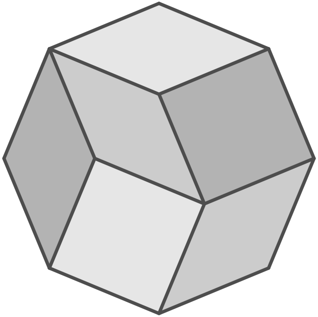
$$\mathrm{drop}(P_1 \cupdot \cdots \cupdot P_n) \geq \min_i \mathrm{drop}(P_i).$$

Proof. (for two polytopes P and Q)

- With $s := \min\{\mathrm{drop}(P), \mathrm{drop}(Q)\}$ and $s' := \mathrm{drop}(P \cupdot Q)$ we have

$$\underbrace{\frac{\overbrace{\mathrm{adj}_P}^{\substack{\leq \\ m_P - d - 1 - s}}}{\prod_{F \subset P} \ell_F}}_{m_P} + \underbrace{\frac{\overbrace{\mathrm{adj}_Q}^{\substack{\leq \\ m_Q - d - 1 - s}}}{\prod_{F \subset Q} \ell_F}}_{m_Q} = \underbrace{\frac{\overbrace{\prod_{F \subset Q} \ell_F \, \mathrm{adj}_P + \prod_{F \subset P} \ell_F \, \mathrm{adj}_Q}^{\substack{\leq \\ (m_P - d - 1 - s) + m_Q \\ = (m_Q - d - 1 - s) + m_P \\ = (m_P + m_Q) - d - 1 - s \\ = (m_{P \cupdot Q} - d - 1 - s) + 2}}}{\prod_{F \subset P} \ell_F \prod_{F \subset Q} \ell_F}}_{m_P + m_Q = m_{P \cupdot Q} + 2} \stackrel{!}{=} \underbrace{\frac{\overbrace{\mathrm{adj}_{P \cupdot Q}}^{m_{P \cupdot Q} - d - 1 - s'}}{\prod_{F \subset P \cupdot Q} \ell_F}}_{m_{P \cupdot Q}}$$

$\Longrightarrow \; s' \geq s$ (Note: proving this projectively is much easier) □



Questions:

- What characterizes the class of polytopes with drop s?
- How to tell the drop of a polytope from geometric/combinatorial characteristics?

Drop is inherited by faces

Lemma.

For a facet F of P holds

$$\mathrm{drop}(F) \geq \mathrm{drop}(P) - 1$$

with equality if and only of P has a facet parallel to F.

Proof.

$$\frac{\overset{m_F-(d-1)-1-s_F}{\mathrm{adj}_F(x)}}{\underset{m_F}{\prod_{G<F}\ell_G(x)}} = \Omega(F;x) = \frac{\overset{\leq m-d-1-s}{\mathrm{adj}_P(x)|_F}}{\underset{m-\begin{cases}2 & \text{has parallel facet}\\ 1 & \text{no parallel facet}\end{cases}}{\prod_{G\neq F}\ell_G(x)|_F}}$$

$\implies s_F \geq s-1$ $\square$

Consequences

Lemma.

A d-polytope has

$$\mathrm{drop}(P) \leq d - 1.$$

Proof.

- $d = 1$: line segment has $\mathrm{drop}([0,1]) = 0$.
- a d-polytope has $\mathrm{drop}(P) \leq \mathrm{drop}(F) + 1$ for each facet F. $\square$

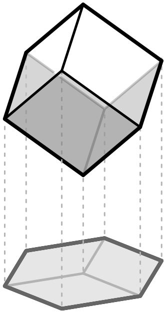
- we already saw that cubes have maximal drop.
- **Question:** which polytopes have maximal degree drop?

Projections

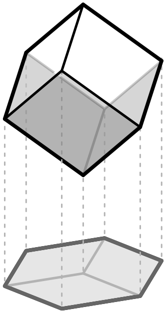
Lemma.

If π is a linear projection onto a $(d-1)$-dimensional subspace, then

$$\operatorname{drop}\big(\pi(P)\big) \geq \operatorname{drop}(P) - 1.$$



Projections, products and sums



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Lemma.

$$\operatorname{drop}(P_1 \times \cdots \times P_n) = \sum_i \operatorname{drop}(P_i) + n - 1$$

$$\operatorname{drop}(P_1 + \cdots + P_n) \geq \sum_i \operatorname{drop}(P_i) + (d-1) - \sum_i (d_i - 1)$$

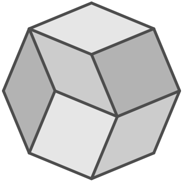
Centrally symmetric polygons

Lemma.

A centrally symmetric polygon P has $\operatorname{drop}(P) = 1$. *(which is maximal)*

Proof I.

- a cs polygon decomposes into parallelograms $\square$



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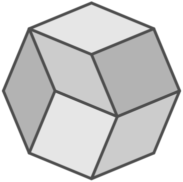

Note: zonotopes also decompose into "skew cubes" (parallelepipedes).

Lemma.

Zonotopes have maximal degree drop $d - 1$.

Centrally symmetric polygons

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Proof II.

- We have $\Omega(P;x) = \Omega(P;-x)$ due to symmetry.
- Since $\Omega = \mathrm{adj}_P \,/ \prod_F \ell_F$, we have adj_P and $\prod_F \ell_F$ both even or both odd.
- Since P is cs, $\deg \prod_F \ell_F = m = 2\bar{m}$ is even.
- Hence $\deg \mathrm{adj}_P = 2\bar{m} - 2 - 1 - \mathrm{drop}(P)$ is even only if $\mathrm{drop}(P) = 1$. $\square$

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Note: Argument applies in all dimensions.

Lemma.

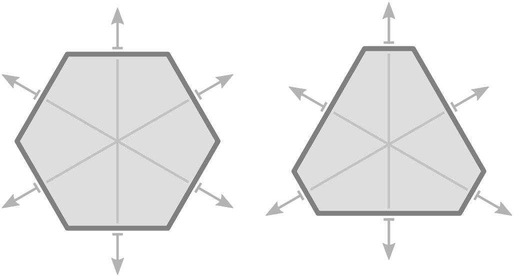
If P is centrally symmetric, then $\deg \mathrm{adj}_P$ is even. In other words

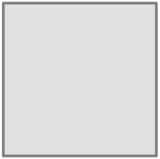
$$\mathrm{drop}(P) \text{ is } \begin{cases} \text{even} & \text{if } d \text{ is odd} \\ \text{odd} & \text{if } d \text{ is even} \end{cases}$$

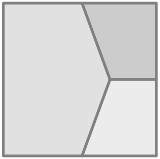
and in particular, cs polytopes in even dimension have $\mathrm{drop}(P) \geq 1$.

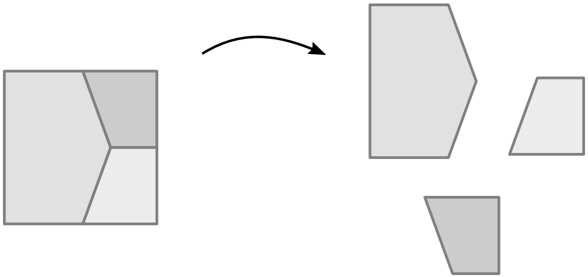
Is there anything else?

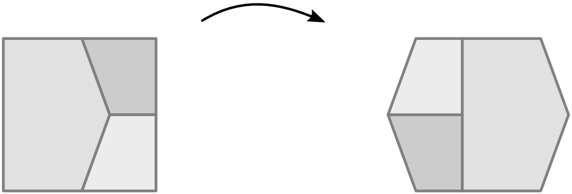
Observation: for maximal drop facets must come in parallel pairs.

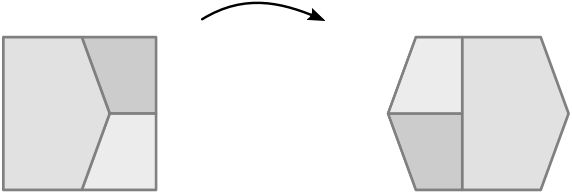












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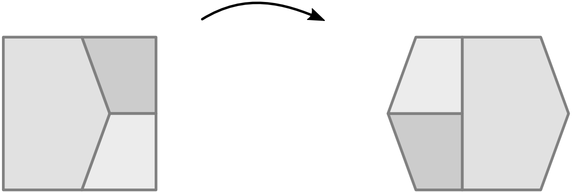

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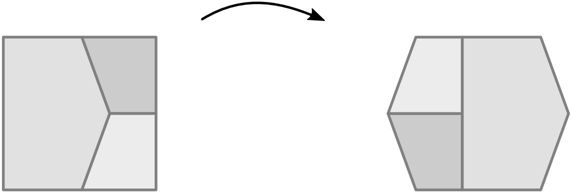
Scissors congruence



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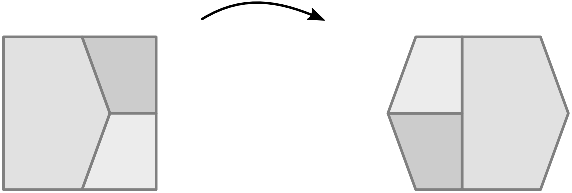
TRANSLATION SCISSORS CONGRUENCE



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Our new favourite (translation-invariant) valuation

$$\Omega_0$$

The view from infinity

$$\Omega(P;x_0,x) := \frac{\mathrm{adj}_P(x_0,x)}{\prod_F \ell_F(x_0,x)} \quad \begin{array}{l} \leftarrow \text{homogenized to degree } m-d-1 \\ \leftarrow \text{homogenized to degree } m \end{array}$$

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{\prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting Ω to the hyperplane at infinite (given by $x_0 = 0$).
- restricting numerator (resp. denominator) to the monomials of degree $m-d-1$ (resp. m).

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Lemma.

Ω_0 *is a translation-invariant valuation.* *(but Ω is not)*

Proof idea. Translation preserve the leading coefficients of a polynomial:

$$p(x) = \sum_n p_n x^n \quad \longrightarrow \quad p(x+t) = \sum_n p_n (x+t)^n.$$

How to use Ω_0

Observation: $\Omega_0(P) = 0$ if and only if $\mathrm{drop}(P) < 0$.

Theorem.

If P and Q are translation scissors congruent, then

$$\mathrm{drop}(P) < 0 \quad \Longleftrightarrow \quad \mathrm{drop}(Q) > 0.$$

But ...

- We can only distinguish drop vs. no-drop.
- We lose all information about the precise value of the degree drop

A note on extension

Note: Ω and Ω_0 are initially defined only on convex polytopes.

Well-known extension theorems apply:

- Ω_0 can be extended to arbitrary unions $P_1 \cup \cdots \cup P_n$
  $\longrightarrow$ non-convex, non-connected, etc.
- Ω_0 can be extended to $\mathbb{Z}$-linear combinations of polytopes
  $\longrightarrow$ weighted polytopes, negative polytopes, etc.
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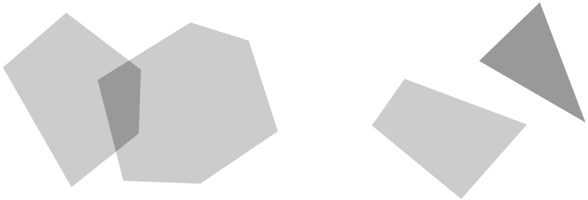

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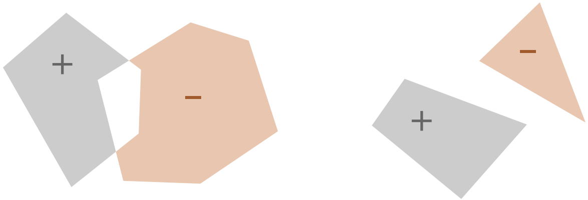


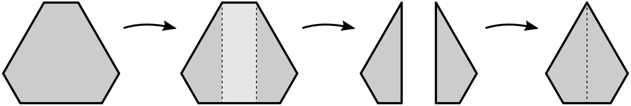
Central symmetry $\Leftrightarrow$ drop $= 1$

Theorem.

For $d = 2$ we have $\mathrm{drop}(P) > 0$ if and only if P is centrally-symmetric.

Proof.

- every edge needs a parallel edge $\implies$ must be a $2n$-gon



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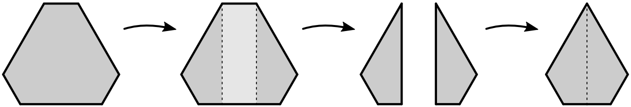


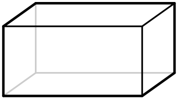
- $\Omega_0(P) = 0$ and this is preserved in all steps ↯ $\square$

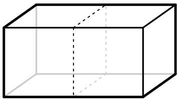
Theorem.

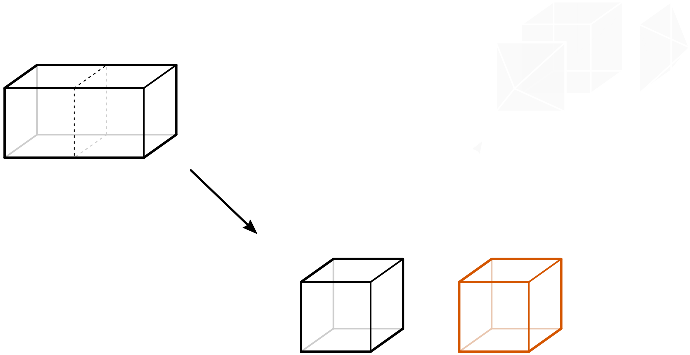
P has maximal degree drop $\mathrm{drop}(P) = d - 1$ iff P is a zonotope.

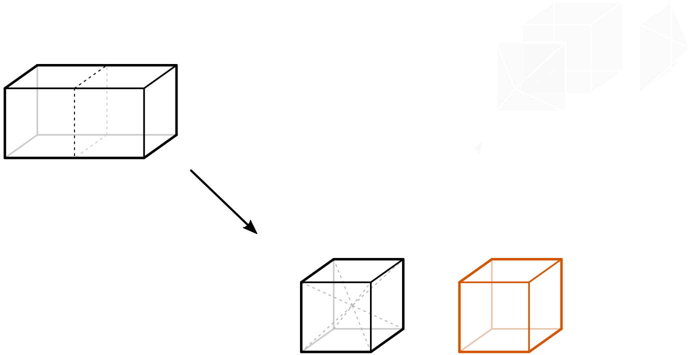
Proof.

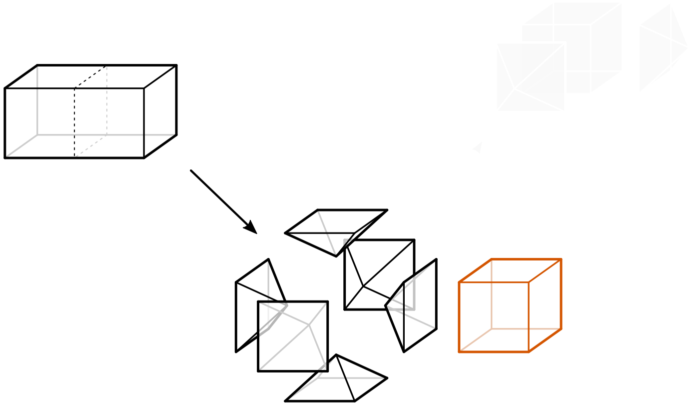
- if P has maximal drop, then so do the faces.
- all faces centrally symmetric $\implies$ zonotope. $\square$

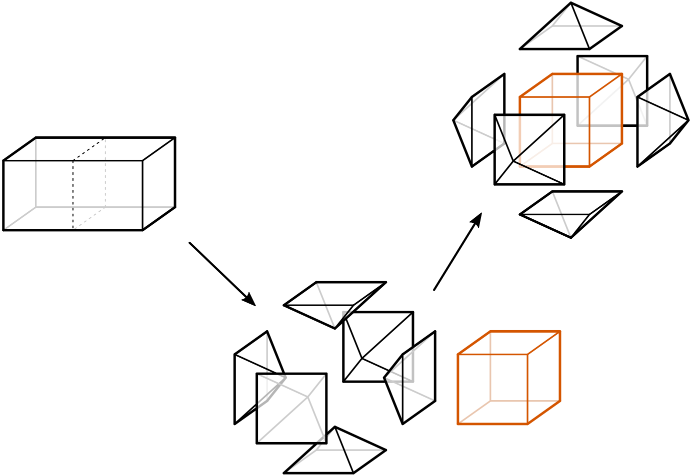


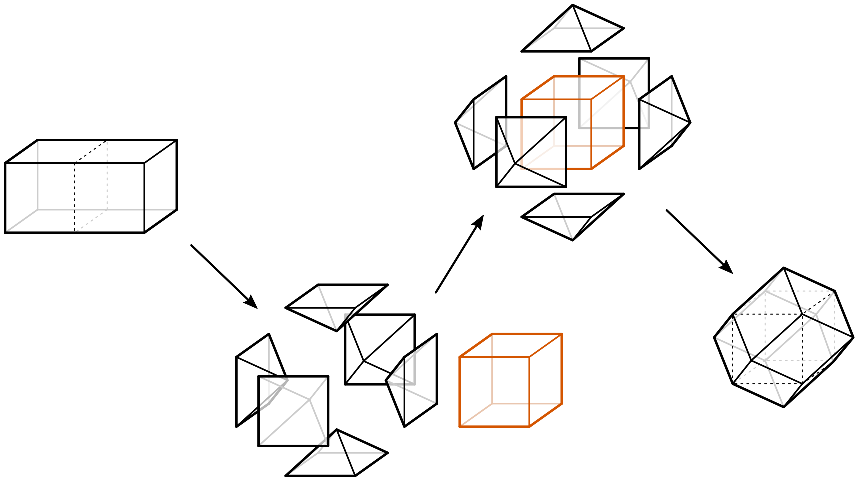


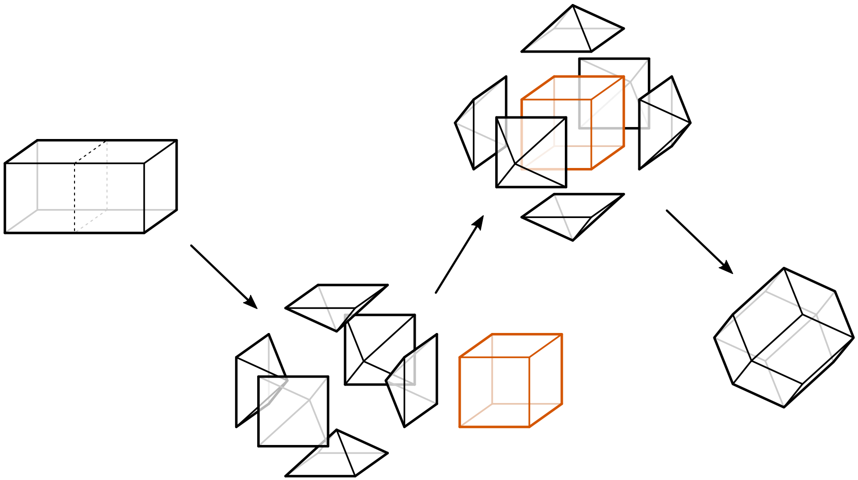


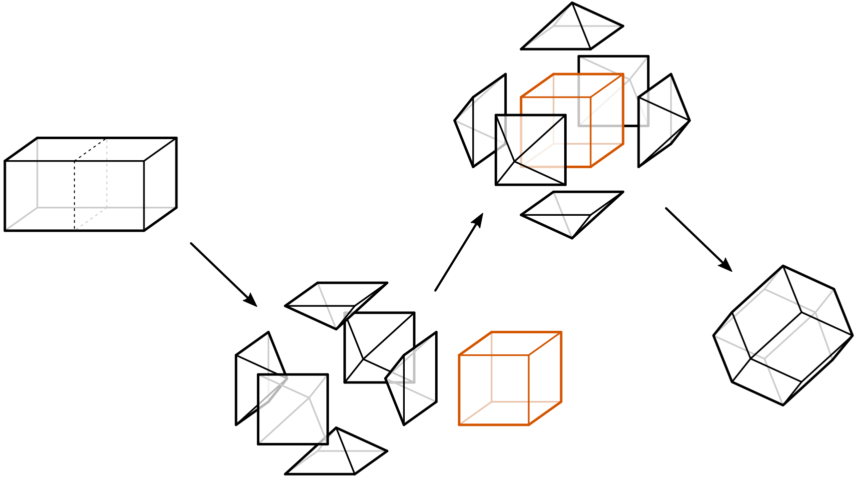












Question: Are zonotopes only translation scissors congruent to zonotopes?
or stronger, is the precise degree drop preserved under TSC?

Yes and no

Theorem.

In dimension $d \leq 3$ translation scissors congruence preserves the degree drop.

Proof. (for $d = 3$)

- if $\mathrm{drop}(P) = 0$ then $\mathrm{drop}(Q) = 0$.
- if $\mathrm{drop}(P) = 2$ then P is a zonotop, hence centrally symmetric. Both $\mathrm{drop} > 0$ and cs are preserved by TSC. But cs 3-polytopes have an even drop. Hence $\mathrm{drop}(Q) = 2$ as well.
- $\mathrm{drop}(P) = 1 \implies \mathrm{drop}(Q) = 1$ follows from $\mathrm{drop} \in \{0, 1, 2\}$. $\square$

This is not true in dimensions $d \geq 4$.

Example: 4-cube and 24-cell.

Homogeneity

Homogeneity of Ω_0

A valuation is ***k*-homogeneous** if for $\lambda > 0$ holds

$$\phi(\lambda P) = \lambda^k \phi(P).$$

Lemma.

Ω_0 is 1-homogeneous. *(but Ω is not)*

Proof.

$$\begin{aligned}\Omega(\lambda P; x) &= \mathrm{vol}(\lambda P - x)^\circ \\ &= \mathrm{vol}(\lambda(P - x/\lambda))^\circ \\ &= \mathrm{vol}(\lambda^{-1}(P - x/\lambda)^\circ) \\ &= \lambda^{-d}\,\mathrm{vol}(P - x/\lambda)^\circ = \lambda^{-d}\,\Omega(P; x/\lambda).\end{aligned}$$

$$\begin{aligned}\Omega_0(\lambda P; x) &= \lambda^{-d}\,\Omega(P; 0, x/\lambda) = \lambda^{-d}\frac{\mathrm{adj}_P(0, x/\lambda)}{\prod_F \ell_F(0, x/\lambda)} \\ &= \lambda^{-d}\frac{\lambda^{-(m-d-1)}\,\mathrm{adj}_P(0,x)}{\lambda^{-m}\prod_F \ell_F(0,x)} = \lambda\frac{\mathrm{adj}_P(0,x)}{\prod_F \ell_F(0,x)} = \lambda\,\Omega_0(P; x).\end{aligned}$$

$\square$

Why homogeneity is great!

Theorem. (McMullen)

If Ω_0 is 1-homogeneous, then it is **Minkowski additive***:*

$$\Omega_0(P_1 + \cdots + P_n) = \Omega_0(P_1) + \cdots + \Omega_0(P_n).$$

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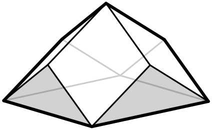
Observation: Minkowski sums of low-dimensional polytopes have a degree drop.



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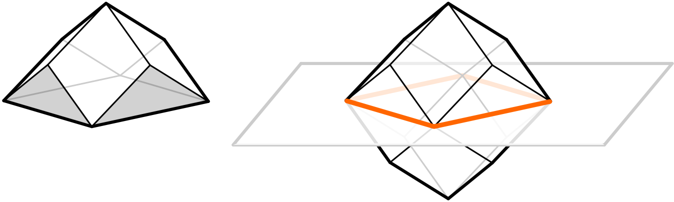

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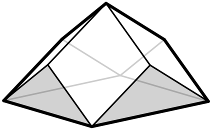

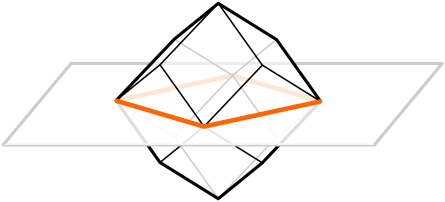

Theorem.

If P is a centrally-symmetric polytope of odd dimension with $\mathrm{drop}(P) > 0$, then each half Q of a central dissection has $\mathrm{drop}(Q) > 0$ as well.

McMullen's decomposition

Theorem. (McMullen)

If Ω_0 is translation-invariant, 1-homogeneous and weakly continuous, then there is a valuation ϕ on $(d-1)$-cones so that

$$\Omega_0(P) = \sum_{e \subset P} \operatorname{len}(e)\phi(N_P(e)).$$

Questions:

- How to verify weak continuity?
- How to determine the valuation ϕ?

McMullen's decomposition for $d = 2$

Theorem.

For $d = 2$ holds

$$\Omega_0(P;x) = -\frac{1}{\|x\|^2}\sum_e \frac{\mathrm{len}(e)}{\langle x, u_e\rangle}.$$

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$$\Omega_0(P; x) = -\frac{1}{\|x\|^2} \sum_e \frac{\mathrm{len}(e)}{\langle x, u_e \rangle}.$$

Case study: the triangle

$$\frac{\mathrm{adj}_\Delta}{\langle x, u_1\rangle\langle x, u_2\rangle\langle x, u_3\rangle} = -\frac{1}{\|x\|^2}\Big(\frac{\ell_1}{\langle x, u_1\rangle} + \frac{\ell_2}{\langle x, u_2\rangle} + \frac{\ell_3}{\langle x, u_3\rangle}\Big)$$

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$$= \frac{\mathrm{Area}(\Delta)}{\mathrm{CircR}(\Delta)}.$$

Open questions

Conjecture.

$$\Omega_0(P;x) = -\frac{1}{\|x\|^2} \sum_e \mathrm{len}(e)\, \Omega\big(T_P(e)\big).$$

Question

How else to characterize polytopes with a fixed degree drop?

Question

What is the relation between Ω_0 and the **Hadwiger invariants**?

Thank you.