Progress on the stress-flex conjecture for coned polytope frameworks

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joined work with Roman Prosanov & Ivan Izmestiev

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Coned polytope frameworks

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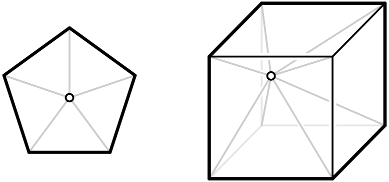
- the skeleton of a convex polytope $P \subset \mathbb{R}^d$
- an interior point (the cone point)
- edges between the cone point and polytope vertices.



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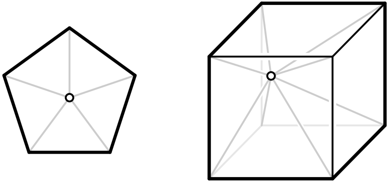

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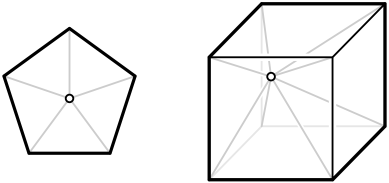


Theorem. (W., 2023)

Coned polytope frameworks are rigid.

$$\#\text{DOFs} - \#\text{constraints} = (\overset{V}{8} + 1) \times \overset{d}{3} - (\overset{E}{12} + \overset{V}{8}) = 7 = 6 + 1.$$

Global and universal rigidity for CPFs

Conjecture.

A CPF is uniquely determined by its graph and edge lengths.

Attention: this is a strong statement!

- we do not input the polytope's combinatorics.
- we do not input the polytope's dimension.

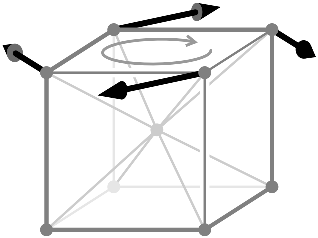
Theorem. (W., 2023)

The conjecture is true

- *for centrally symmetric CPFs.*
- *for given combinatorial type.*

First-order theory of CPFs

Simple CPFs (i.e. vertex degree $= d$) are essentially never first-order rigid:



$$\color{red}{\#\text{DOFs}} - \color{blue}{\#\text{constraints}} = \color{red}{d(|V|+1)} - \color{blue}{(\underbrace{|E|}_{=d/2|V|}+|V|)} = \dots = (d/2-1)|V| + d.$$

$$\overset{?}{\geq} \#\text{trivial flexes}$$

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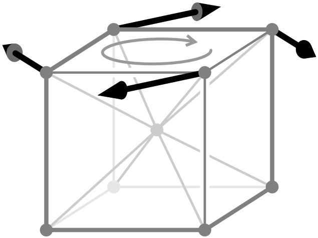

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first-order rigid ✗ $\implies$ prestress stable $\implies$ second-order rigid $\implies \cdots \implies$ rigid ✓

Second-order theory

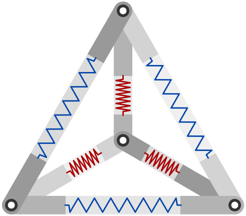
Second-order primer

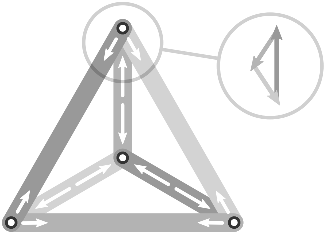
One aims to show that no first-order flex "becomes real":

- **Second-order rigid** := every first-order flex $\dot{\boldsymbol{p}}$ is **blocked** by some stress $\boldsymbol{\omega}$:

$$\sum_{vw \in E} \omega_{vw} \|\dot{p}_v - \dot{p}_w\|^2 \neq 0.$$

- **Prestress stable** (PSS) := there is a single stress $\boldsymbol{\omega}$ that blocks every first-order flex.





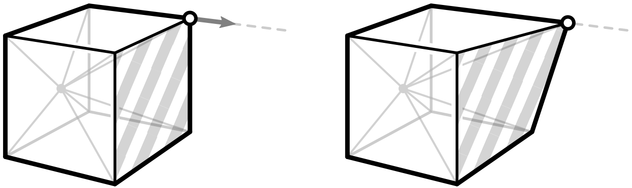
Second-order theory for CPFs

Conjecture

CPFs are prestress stable.

... and we know exactly which stress to pick: the **Wachspress stress**

This stress starts to exist only when all faces become flat:



THE WACHSPRESS-IZMESTIEV STRESS

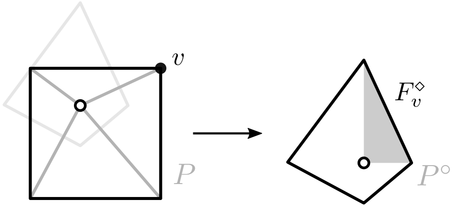
The **Wachspress-Izmestiev stress** $\boldsymbol{\omega}^{\mathrm{W}}$ exists for every CPF:

$\omega^{\mathrm{W}}_{v\star} = \omega^{\mathrm{W}}_{v} =$ **Wachspress coordinate** of the cone point at vertex v

$\omega^{\mathrm{W}}_{vw} = vw$-entry of **Izmestiev matrix**

$$\omega^{\mathrm{W}}_{v} = \frac{\mathrm{vol}(F^{\diamond}_{v})}{\|p_i\|}, \qquad \omega^{\mathrm{W}}_{vw} = \frac{\mathrm{vol}(F^{\diamond}_{v} \cap F^{\diamond}_{w})}{\|v\|\|w\| \sin \sphericalangle(v,w)}.$$

For *simple* CPFs it is the only stress.



The Stress-Flex Conjecture

A helpful/mysterious observation

We want: for all first-order flexes $\dot{\boldsymbol{p}}\colon V \to \mathbb{R}^d$ holds (fixing $p_\star = \dot{p}_\star = 0$)

$$\sum_{v,w} \omega^{\mathrm{W}}_{vw} \|\dot{p}_v - \dot{p}_w\|^2 + \sum_v \omega^{\mathrm{W}}_v \|\dot{p}_v\|^2 > 0$$

It turned out it would suffice to show the following:

$$\sum_v \omega^{\mathrm{W}}_v \, \dot{p}_v = 0.$$

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The (weak) stress-flex conjecture (Connelly, Gortler, Theran, W.)

Given a CPF with Wachspress stress $\boldsymbol{\omega}^{\mathrm{W}}$. For each first-order flex $\dot{\mathbf{p}}$ holds

$$\sum_{v} \omega^{\mathrm{W}}_{v} \dot{p}_v = 0. \quad \longleftarrow \text{ stress-flex orthogonality}$$

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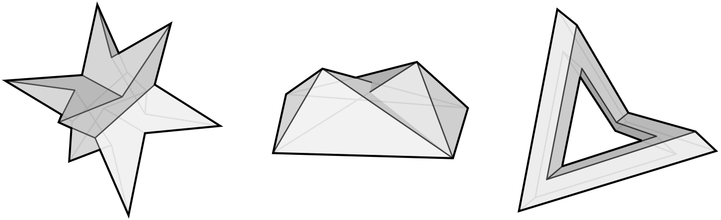

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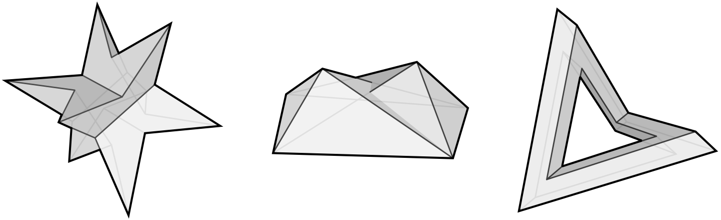
Conclusion: might be less about polytopes and more about *closed PL-surfaces*.



The full conjecture

The stress-flex conjecture

Let $S \subset \mathbb{R}^d$ be a closed PL-surface and consider its the coned skeleton (aka a CSF). If $\dot{\boldsymbol{p}}$ is a first-order flex and $\boldsymbol{\omega}$ is a stress, then

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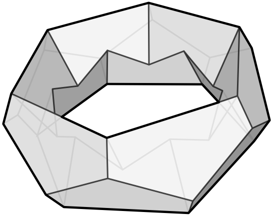

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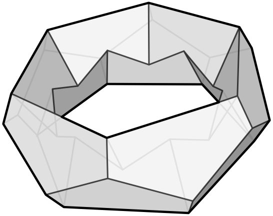
Question: Does stress-flex orthogonality ever <u>not</u> hold?



Stress-flex orthogonality holds generically

Observation (Dewar)

For a generic coned framework for any first-order flex $\dot{\boldsymbol{p}}$ and stress $\boldsymbol{\omega}$ holds:

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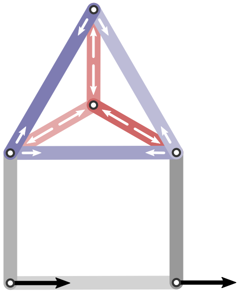
Intuition:

- stresses and flexes live on different parts of a framework.



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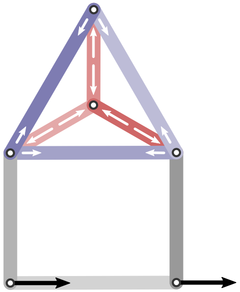
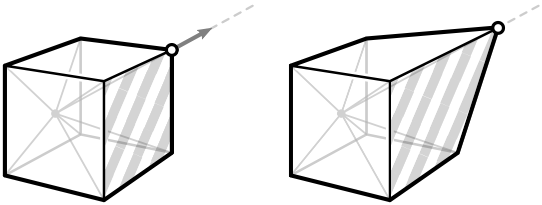
But ... CPFs are very non-generic

Better question:

- Why does stress-flex orthogonality still hold?
- Where else do stresses/flexes coexist?



Non-example I



Lemma.

First-order flexes and stresses of coned frameworks are preserved by moving vertices radially.

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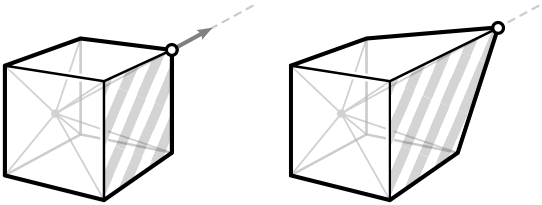


Lemma.

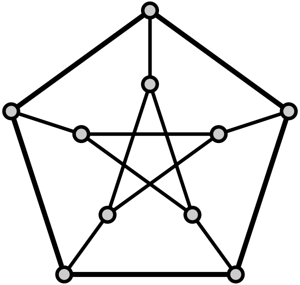
First-order flexes and stresses of coned frameworks are preserved by moving vertices radially.

Observation: Moving vertices radially destroys flex-stress orthogonality.

Non-example II

Spectral embeddings of sparse graphs have stresses and flexes!

... in fact, CPFs are spectral embeddings (Izmestiev, 2007)



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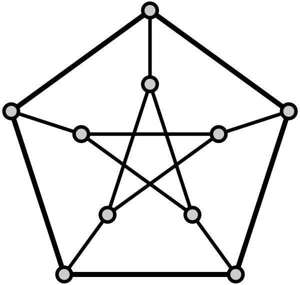

Observation: General spectral embeddings do *not* satisfy stress-flex orthogonality.

... e.g. 4- and 5-dimensional embeddings of Petersen graph.

A Stoker type Conjecture

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$P(t)$ ... differentiable family of polytopes (or any *orientable* surface)
$n_F(t)$ ... normal of facet F
$V_F(t)$ ... volume of facet F

Minkowski's
balancing condition

$$0 = \sum_F V_F n_F$$

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$$0 = \sum_F V_F n_F \quad \Longrightarrow \quad 0 = \frac{\mathrm{d}}{\mathrm{d}t} \sum_F V_F n_F = \sum_F \dot{V}_F n_F + \sum_F V_F \dot{n}_F$$

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$P(t)$ ... differentiable family of polytopes (or any *orientable* surface)
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$\theta_{FG}(t)$ ... dihedral angle between facet F and G

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Conjecture

Suppose $\dot{\theta}_{FG} = 0$ *whenever* F *and* G *are incident at* $t = 0$. *Then*

$$\sum_F \dot{V}_F n_F = \sum_F V_F \dot{n}_F = 0.$$

Our progress

We solved ...

- the Stoker type conjecture in 3D
- the stress-flex conjecture for the Wachspress stress in 3D
- prestress stability of CPFs in 3D.

Some words on the proof …

We have $\frac{\mathrm{d}}{\mathrm{d}t}\langle n_F, n_G\rangle = 0$. We prove

$$\sum_F V_F \dot{n}_F = \sum_F \dot{V}_F n_F = 0.$$

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Three ingredients

$$0 = \sum_{v:v\sim F} \dot{\alpha}_{Fv} \qquad \longleftarrow \quad \text{angle sum in } n\text{-gon is } \pi(n-2)$$

$$0 = \sum_{F:F\sim v} \dot{\alpha}_{Fv} n_F \qquad \longleftarrow \quad \text{well-known argument from spherical geometry}$$

(this uses $\dot{\theta}_{FG} = 0$)

$$2\dot{V}_F = \sum_{v:v\sim F} \dot{\alpha}_{Fv} h_{Fv}^2 \qquad \longleftarrow \quad \text{a medium long computation}$$

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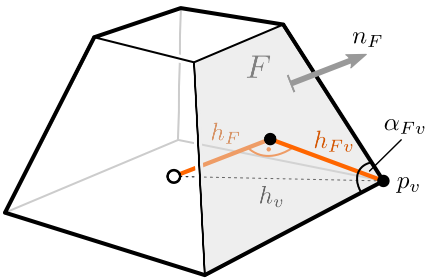


Some words on the proof ...

We use

$$0 = \sum_{v:v\sim F} \dot\alpha_{Fv} \qquad 0 = \sum_{F:F\sim v} \dot\alpha_{Fv} n_F \qquad 2\dot V_F = \sum_{v:v\sim F} \dot\alpha_{Fv} h_{Fv}^2$$

to establish

$$\begin{aligned}\dot V_F = \tfrac12 \sum_{v:v\sim F} \dot\alpha_{Fv} h_{Fv}^2 &= \tfrac12 \sum_{v:v\sim F} \dot\alpha_{Fv}(h_v^2 - h_F^2) \\ &= \tfrac12 \sum_{v:v\sim F} \dot\alpha_{Fv} h_v^2 - \tfrac12 h_F^2 \overbrace{\sum_{v:v\sim F} \dot\alpha_{Fv}}^{=0} = \tfrac12 \sum_{v:v\sim F} \dot\alpha_{Fv} h_v^2\end{aligned}$$

$$\sum_F n_F \dot V_F = \tfrac12 \sum_F n_F \sum_{v:v\sim F} \dot\alpha_{Fv} h_v^2 = \tfrac12 \sum_v h_v^2 \underbrace{\sum_{F:F\sim v} \dot\alpha_{Fv} n_F}_{=0} = 0$$

Consequences

Theorem.

The stress-flex conjecture holds for the Wachspress stress and $d = 3$.

Proof.

- S orientable: immediate from Stoker type result
- S non-orientable: double cover the surface; it becomes orientable; apply Stoker type result. □

Theorem.

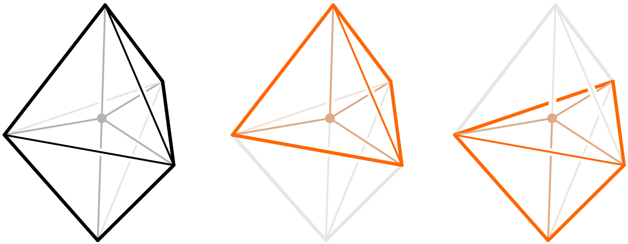
Coned polytope frameworks for $d = 3$ are prestress stable.

What remains …

What about other stresses?

There are at least two potential approaches to this:

- Maybe all other stresses are generic (à la Dewar).
- Maybe all stresses are Wachspress stresses in some sense



What about higher dimensions?

F ... dimension 2
$F \ni v$ ... dimension 0

$$0 = \sum_{G:G\subset F} \dot{\alpha}_{FG}$$

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$F \ni v$ ... dimension 0 = codimension 3 $\quad$ $F \supset G$ ... codimension 3

$$0 = \sum_{G:G\subset F} \dot\alpha_{FG}\,\mathrm{vol}(G)$$ $\longleftarrow$ Schläfli formula

$$0 = \sum_{F:F\supset G} \dot\alpha_{FG} n_F$$ $\longleftarrow$ same as before (uses $\dot\theta_{k\ell} = 0$)

$$2\dot V_F = \sum_{G:G\subset F} \dot\alpha_{FG} h_{FG}^2\,\mathrm{vol}(G)$$ $\longleftarrow$???

Thank you.

- M. Winter, *"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"* (2023)
- R. Connelly, S. J. Gortler, L. Theran, M. Winter, *"Energies on Coned Convex Polytopes"* (2024)
- R. Connelly, S. J. Gortler, L. Theran, M. Winter, *"The Stress-Flex Conjecture"* (2024)