# The structure of 2-level polytopes

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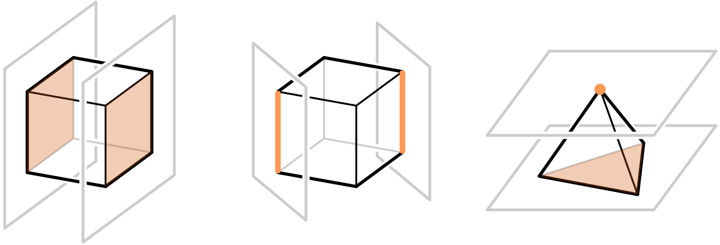
# 2-LEVEL POLYTOPES

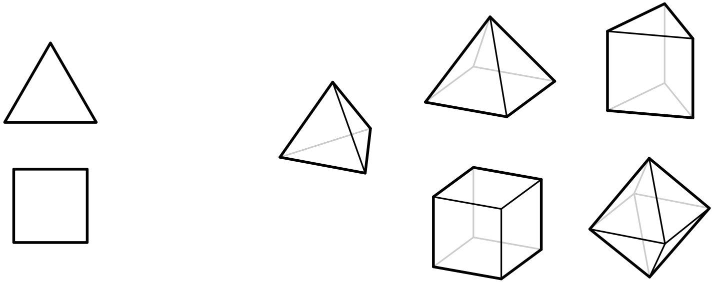
$$P = \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d,\ d \geq 0$$

**Definition.**

- Two faces $F_1, F_2 \subseteq P$ are **antipodal** if they are contained in parallel hyperplanes (*i.e.* there are parallel hyperplanes $H_1, H_2 \subseteq \mathbb{R}^d$ with $F_i = P \cap H_i$)
- A polytope $P$ is **2-level** if each antipodal face pair that contains a facet also contains all vertices.



# Examples



| dim | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 2-level | 1 | 1 | 2 | 5 | 19 | 106 | 1150 | 27291 | 1378453 |

# Examples

Many 2-level polytopes are constructed from combinatorial objects:

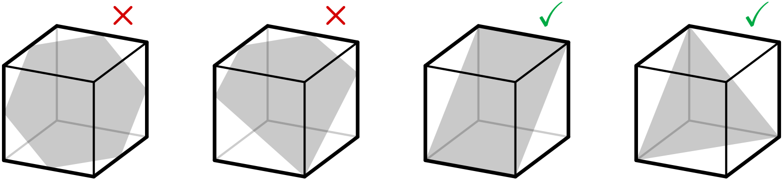
- *Hanner polytopes* (in relation to cographs)
- order polytopes of posets
- stable set polytopes of perfect graphs
  $+$ their twisted prisms ($=$ *Hansen polytopes*)
- spanning tree polytopes of series-parallel graphs
- *Birkhoff polytopes* (from double stochastic matrices)
- certain matroid base polytopes

# Properties

- all faces are 2-level
- closed under products and joins
- #vertices · #facets $\leq d2^{d+1}$
- are 01-polytopes (if $P$ is $d$-dimensional then $P \subseteq [0,1]^d$)

**Theorem.**

*2-level polytopes are precisely the polytopes that can be written as the intersection of a cube with an affine subspace that is spanned by vertices of the cube.*



# Binary scalar products

Based on empirical evidence it was conjectured that

$$f_0(P) \cdot f_{d-1}(P) \le d2^{d+1}.$$

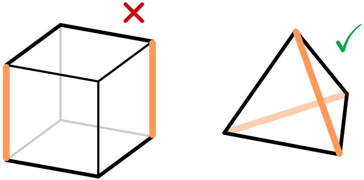
This was eventually proven using **binary scalar product pairs**: $A, B \subset \mathbb{R}^d$

$$\langle a, b\rangle \in \{0, 1\}, \quad \text{for all } a \in A \text{ and } b \in B.$$

**Theorem.** (Kupavskii, Weltge; 2020)

*If $A, B \subseteq \mathbb{R}^d$ is a binary scalar product pair, then $|A| \cdot |B| \le (d+1)2^d$.*

# Binary scalar products



**Theorem.** (Kupavskii, Weltge; 2020)

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$\mathcal{A}_s(P)$ ... set of spanning antipodal face pairs

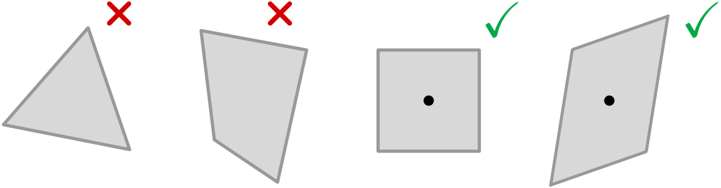
**Corollary.**

*For a 2-level polytope holds $f_0(P) \cdot |\mathcal{A}_s(P)| \leq (d+1)2^d$.*

# Conjectures for centrally symmetric polytopes

# Centrally symmetric polytopes

centrally symmetric $:\Longleftrightarrow \quad P = -P$



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|---|---|---|---|---|---|---|---|---|---|
| 2-level | 1 | 1 | 2 | 5 | 19 | 106 | 1150 | 27291 | 1378453 |
| cs 2-level | 1 | 1 | 1 | 2 | 4 | 13 | 45 | 238 | 1790 |

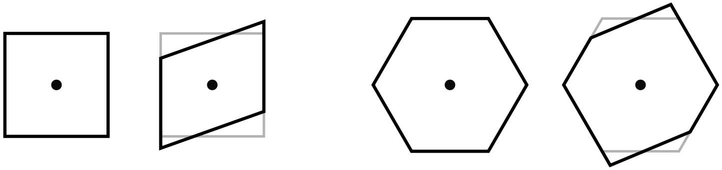
# Properties

- each facet contains exactly half of the vertices
- each vertex lies in exactly half of the facets

  } defining 2-level among cs

  $\Longrightarrow$ closed under polar duality
- precisely the section of a cube with linear space spanned by vertices
- linearly unique (all cs 01-polytopes are linearly unique)

  $\Longrightarrow$ linearly invariant geometric properties are combinatorial properties



# Kalai's $3^d$ conjecture

$$s(P) := \cancel{f_{-1}} + f_0 + f_1 + \cdots + f_{d-1} + f_d = \#\underline{\text{non}}\text{-empty faces}$$

**Conjecture.** ($3^d$ conjecture, Kalai, 1989)

For every centrally symmetric $d$-polytope $P \subset \mathbb{R}^d$ holds

$$s(P) \geq s(d\text{-cube}) = 3^d.$$

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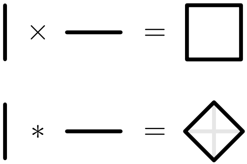
**But:** cube is not the only minimizer! $\rightarrow$ **Hanner polytopes**

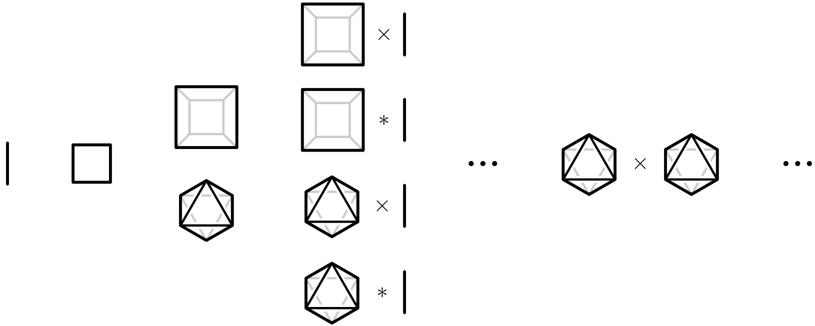
# Hanner polytopes

**Hanner polytopes** are defined recursively:

(i) start from a line segment.
(ii) recursively apply Cartesian products ($\times$) and sums ($*$)



# Hanner polytopes




#Hanner polytopes for $d \geq 1 \ = \ 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, ...$

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**But:** cube is not the only minimizer! $\rightarrow$ **Hanner polytopes**

**What is known ... ?**

- dimension $d \leq 3$ ✓ easy
- dimension $d = 4$ ✓ not so easy (2007)
- simple/simplicial polytopes ✓ needs a lot of algebra
- without requiring central symmetry ✓ easy $\rightarrow$ $s(d\text{-simplex}) = 2^d - 1$

# Mahler's conjecture

**Mahler volume** ... $M(P) := \mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ)$

**Conjecture.** ($3^d$ conjecture, Mahler, 1939)

For every centrally symmetric $d$-polytope $P \subset \mathbb{R}^d$ holds

measures "roundness" $\longrightarrow$ $M(P) \geq M(d\text{-cube}) = \dfrac{4^d}{d!}.$

**But:** cube is not the only minimizer! $\rightarrow$ **Hanner polytopes**

**What is known ... ?**

- dimension $d \leq 3$ ✓ not so easy ($d = 2$: 1939, $d = 3$: 2020)
- dimension $d = 4$ ? out of reach
- Hanner polytopes are local minimizers ✓(2014)
- without requiring central symmetry ? open $\rightarrow M(d\text{-simplex}) = \frac{(d+1)^{d+1}}{(d!)^2}$

# Kalai's flag conjecture

$$S(P) := \#\text{flags of } P$$

**Conjecture.** (flag conjecture, Kalai, 1989)

For every centrally symmetric $d$-polytope $P \subset \mathbb{R}^d$ holds

$$S(P) \geq S(d\text{-cube}) = d!\,2^d.$$

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- verified for $d \leq 7$ by enumeration

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- verified for $d \leq 7$ by enumeration
- 2-level polytopes are "very small polytopes"

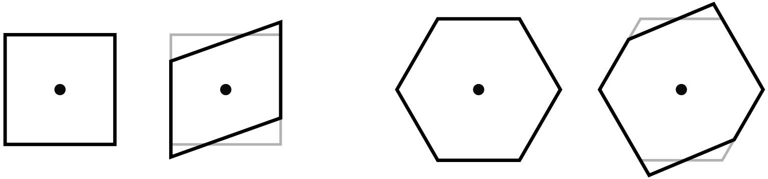
# Linearly unique polytopes

Let $\mathrm{Real}(P)$ be the space of cs realization of $P$ module linear transformations.

**Definition.**

A centrally symmetric polytope is

- **linearly unique** if $\mathrm{Real}(P)$ consists of a single point.
- **linearly discrete** if $\mathrm{Real}(P)$ consists of finitely many points.
- **linearly compact** if $\mathrm{Real}(P)$ is compact.



# Linearly compact polytopes

**Lemma.**

*If $P$ is a cs minimizer of face number, then $P$ is linearly compact.*

*Proof sketch.*

- if $P$ is not linearly compact, then there is a convergent sequence $P_1, P_2, P_3, \ldots$ of realizations of $P$ with $\lim P_n$ not being a realization of $P$.
- observe that in the limit, there cannot be new faces, but faces must have vanished.

$\implies$ $P$ cannot have been a minimizer. □

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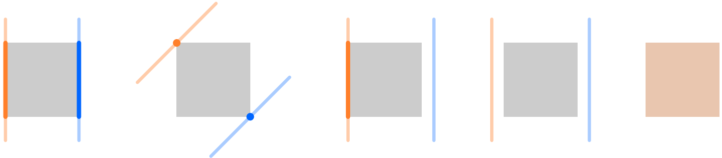
**Conjecture.**

*The only polytopes with compact realization spaces are linearly discrete.*

- true for matroids and oriented matroids.
- polytope realization spaces are unions of oriented matroid realization spaces.

# Kalai for non-cs polytopes

- By convention, say that $(P, \varnothing)$ is an antipodal face pair as well.



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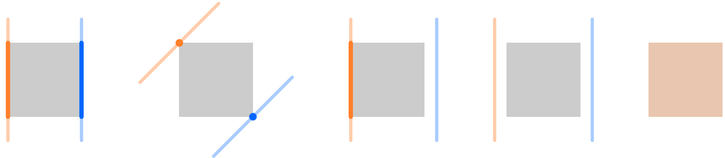

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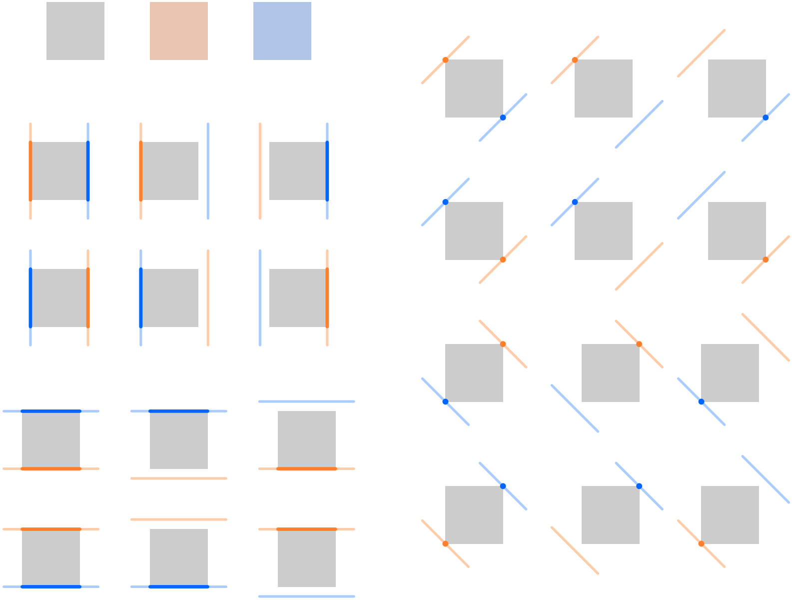
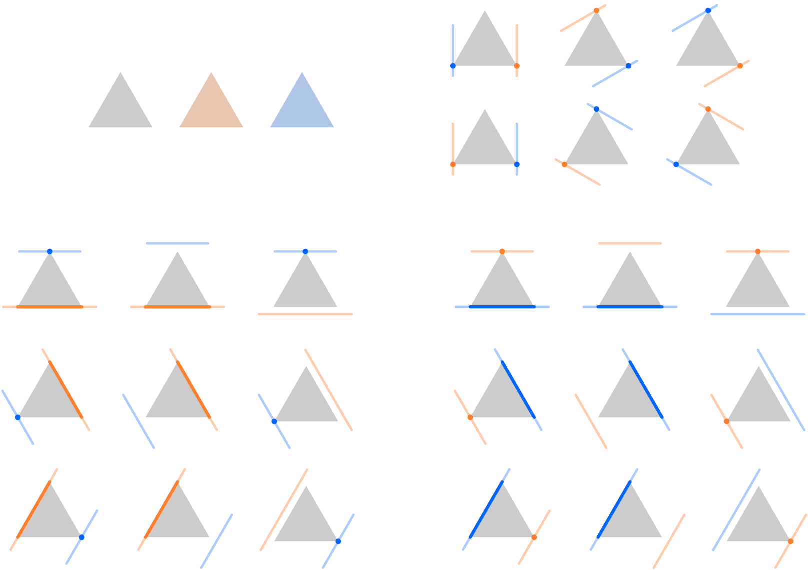
**Conjecture.**

*If $P$ is a (2-level) polytope, then*

$$|\mathcal{A}(P)| \geq 3^{d+1}$$

$$|\mathcal{A}^*(P)| \geq \frac{3^{d+1}+1}{2} = 3 \cdot \frac{3^d - 1}{2} + 2$$





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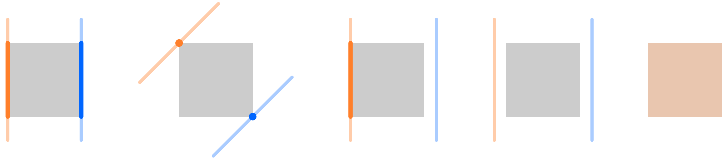

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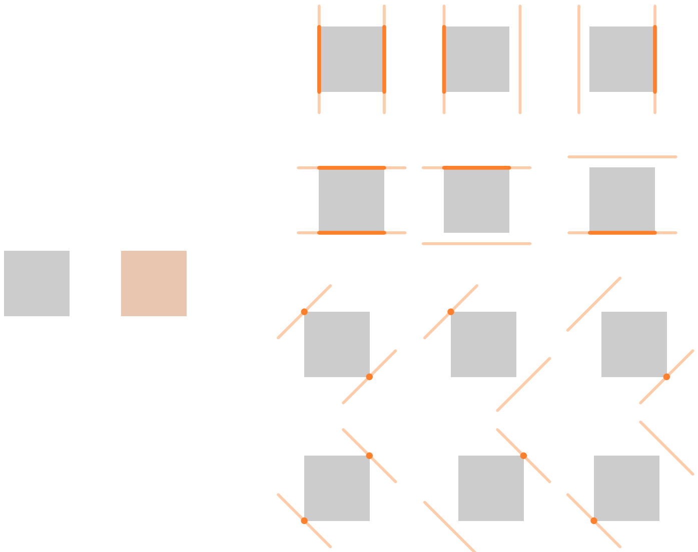
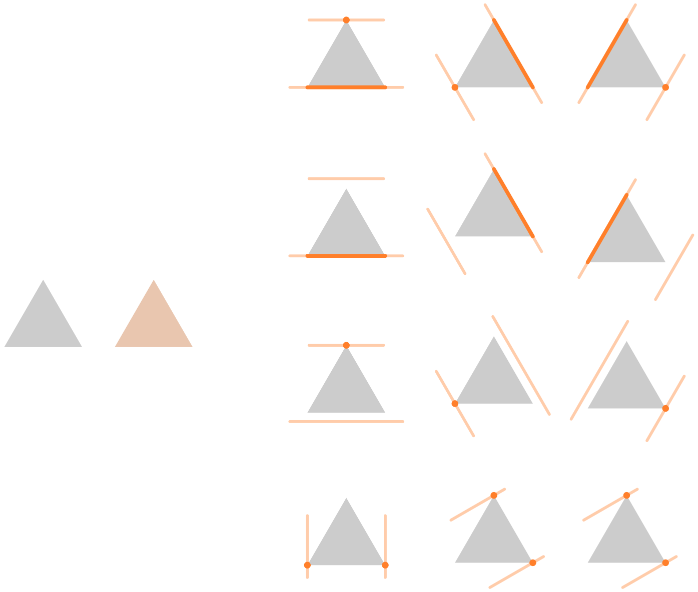
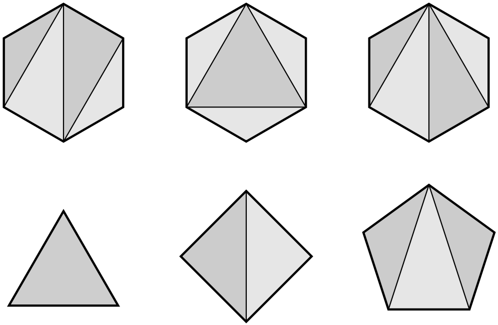




# Mahler's conjecture

$$\mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ) \geq \frac{4^d}{d!}$$

# Volume and triangulation



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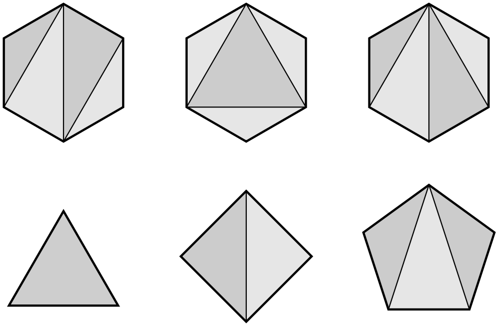

**Theorem.**

*In a 2-level polytopes ...*

- *each simplex in a pulling triangulation has the same volume.*
  *(lattice volume 1)*

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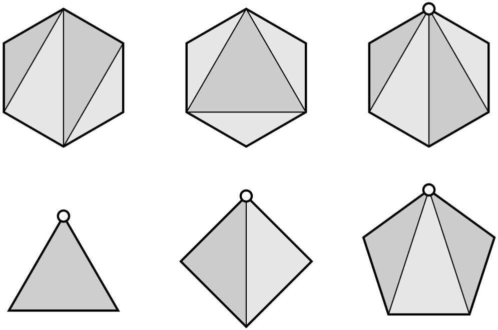

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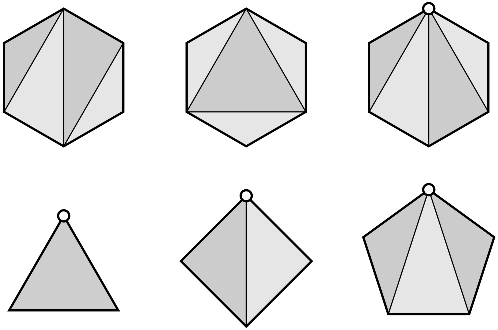

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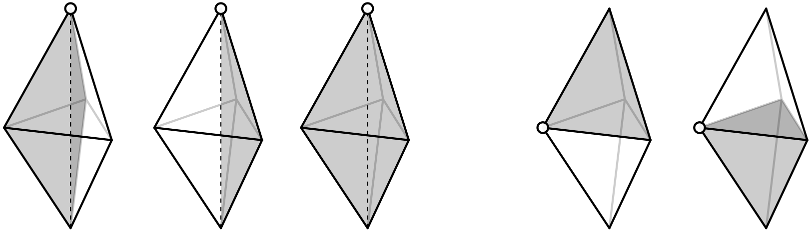

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# Mahler volume and pulling triangulations

$f_d^*(P)$ ... # simplices in pulling triangulation of $P$

The Mahler conjecture is equivalent to the following:

**Conjecture.**

*For a centrally-symmetric 2-level polytope $P \subset \mathbb{R}^d$ holds*

$$f_d^*(P) \cdot f_d^*(P^\circ) \geq d! 2^{d-1}$$

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$$f_d^*(P) \cdot f_d^*(P^\circ) \;\geq\; \tfrac{1}{2}S(P) \;\underset{\substack{\uparrow \\ \textit{flag conjecture}}}{\geq}\; \tfrac{1}{2}S(d\text{-cube})$$

# Pulling triangulations

**Theorem.**

*If $P$ is a 2-level polytope, then*

$$f_d^*(P) = \sum_{F_0 \preceq \cdots \preceq F_d} \left(1 - \frac{f_0(F_0)}{f_0(F_1)}\right) \cdots \left(1 - \frac{f_0(F_{d-1})}{f_0(F_d)}\right).$$

*Proof.*

- For a vertex $v$ and facet $F_{d-1}$, let $[v \notin F_{d-1}]$ denote the indicator function.
- We have
$$f_d^*(P) = \sum_{F_{d-1}} [v \notin F_{d-1}] f_{d-1}^*(F_{d-1}).$$
- Take expectation value *w.r.t.* a uniform random choice of $v$:

$$f_d^*(P) = \sum_{F_{d-1}} \left(1 - \frac{f_0(F_{d-1})}{f_0(P)}\right) f_{d-1}^*(F_{d-1}) = \sum_{F_0 \preceq \cdots \preceq F_d} \left(1 - \frac{f_0(F_0)}{f_0(F_1)}\right) \cdots \left(1 - \frac{f_0(F_{d-1})}{f_0(F_d)}\right)$$

□

# Strongly 2-level polytopes

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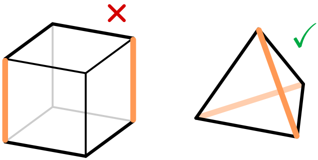
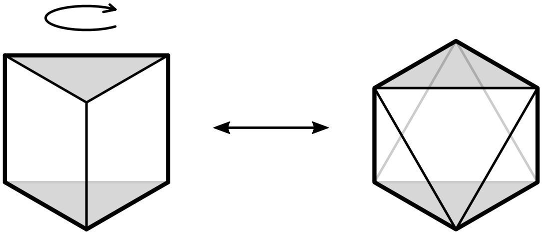
**Definition.**

- An antipodal face pair is **spanning** if its affine span is $\mathbb{R}^d$.
- A polytope $P$ is **strongly 2-level** if each spanning antipodal face pair contains all vertices.

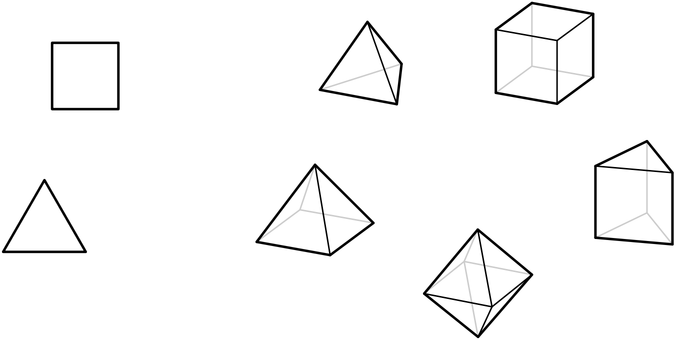


- $d \leq 4$: every 2-level polytope is strongly 2-level.
- $d = 5$: there is a unique 2-level polytope that is not strongly 2-level (the (6,2)-hypersimplex)
- $d \leq 5$: faces of strongly 2-level polytopes are strongly 2-level.
- $d = 6$: there is a unique strongly 2-level polytope with a facet that is not. (the (7,3)-hypersimplex with a (6,2)-hypersimplex as a facet)

# Twisting



# Twisting classes



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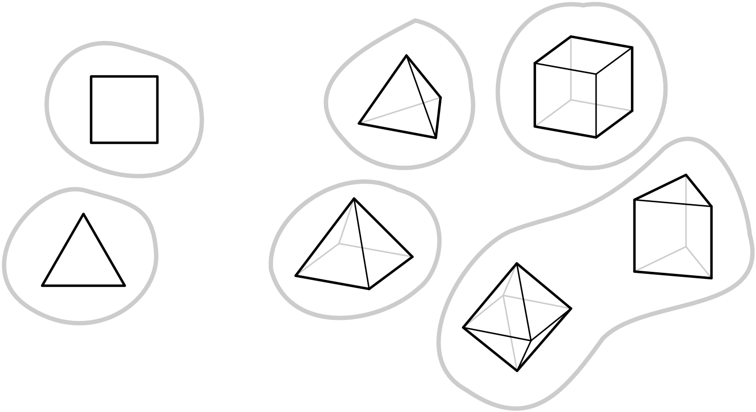
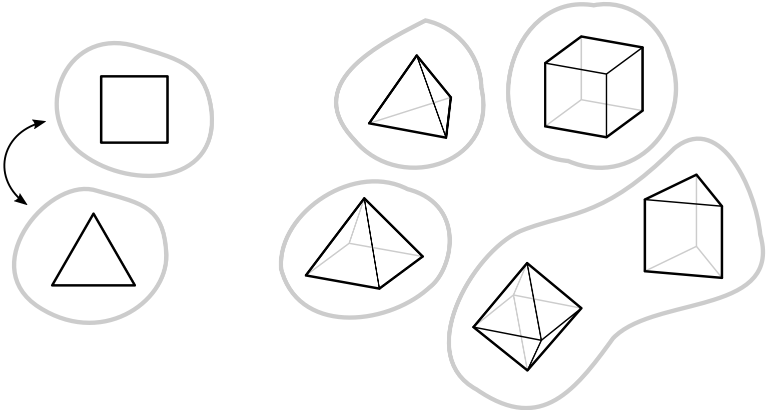


# Twisting classes & twisting duality



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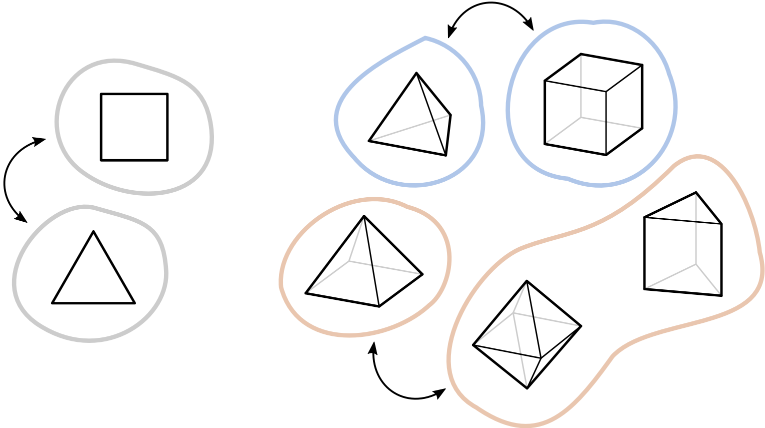

# Twisting duality

All members of a twisting class share

- # vertices
- # vertex orbits
- # antipodal face pairs
- # spanning antipodal face pairs

Within a twisting class one can get from anywhere to anywhere by a single twist.

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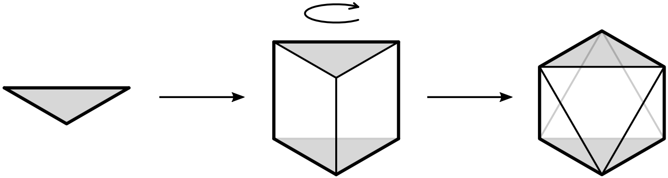
Twisting duality swaps:

- # vertex orbits $\leftrightarrow$ size of twisting class
- # vertices $\leftrightarrow$ # spanning antipodal face pairs

Twisting duality preserves:

- # antipodal face pairs

# Twisted prisms

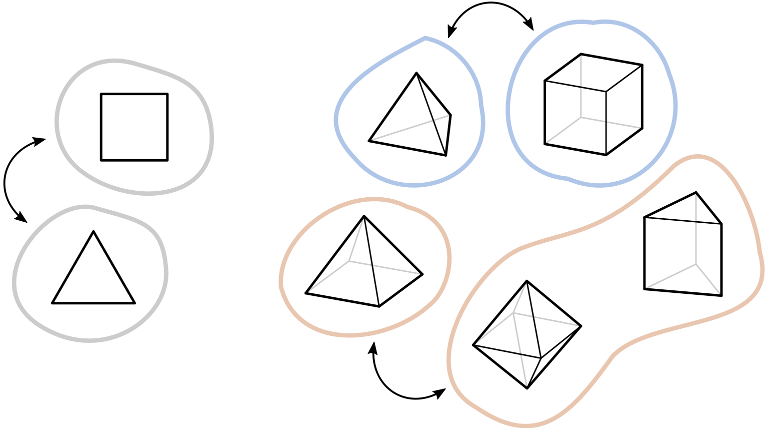


**Lemma.**

*The twisted prism $P \boxminus P$ is 2-level if and only if $P$ is strongly 2-level.*

| $P \boxminus P$ | $P$ |
|---:|---|
| faces | antipodal face pairs |
| facets | spanning antipodal face pairs |
| polar duality | twisting duality |
| # facet types | size of twisting class |

# Twisting classes



# Thank you.