Adjoint degrees and scissors congruence for polytopes

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MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES

combinatorial synergies

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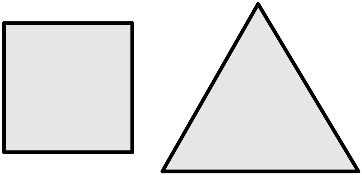
Scissors congruence

Two polytopes P and Q are **scissors congruent** if

$$P = P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n \qquad Q = Q_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} Q_n.$$

with $Q_i = S_i(P_i)$, where $S_i \in \mathrm{Iso}(\mathbb{R}^d)$ are isometries.



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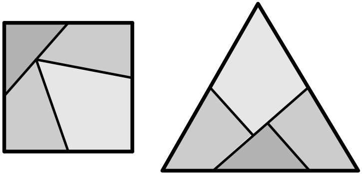

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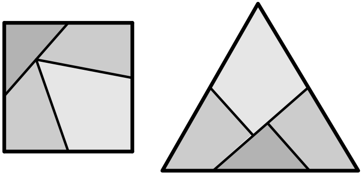

Theorem (Wallace, Bolyai, Gerwien; 1807/33/35)

Two polygons P, Q are scissors congruent if and only if $\mathrm{vol}(P) = \mathrm{vol}(Q)$.

Hilbert's Third Problem

Given any two polyhedra P and Q of equal volume, is it always possible to dissect P into finitely many polyhedral pieces $P_1, ..., P_n$, which can then be reassembled to yield Q?

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Theorem. (Dehn; 1901)

If $P, Q \subset \mathbb{R}^3$ are scissors congruent, then they have the same Dehn invariant.

$$D(P) := \sum_{e \subset P} \ell_e \otimes_{\mathbb{Z}} \theta(e)/2\pi \;\in\; \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}.$$

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Theorem. (Sydler; 1965)

$P, Q \subset \mathbb{R}^3$ are scissors congruent if and only if they have the same volume and the same Dehn invariant.

Valuations

Whenever P, Q, $P \cap Q$ and $P \cup Q$ are polytopes, a **valuation** satisfies

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$$

Examples:

- volume
- Dehn invariant
- surface area measure
- Euler characteristic
- mixed volumes
- number of contained lattice points

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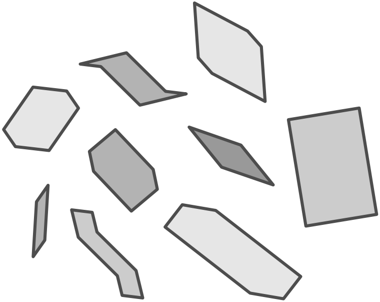
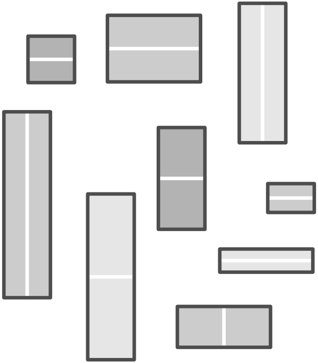
What we mainly care about (true for *simple valuations*):

$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

Two composition puzzles





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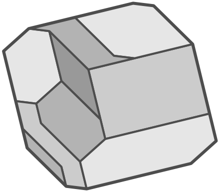

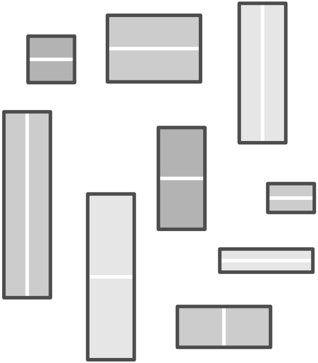

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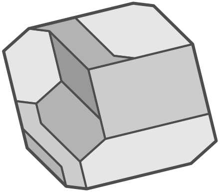
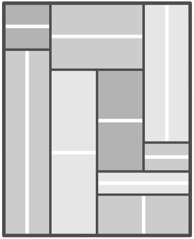
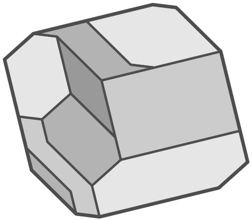




Puzzle I

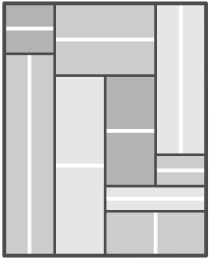


Let $\nu(P)$ be the *surface area measure* of $P \subset \mathbb{R}^d$ on $\mathbb{S}^{d-1}$.

$$\phi(P) := \nu(P) - \nu(-P)$$

Fact: a convex polygon P is centrally symmetric if and only if $\phi(P) = 0$.

Puzzle II



$$\phi(P) := \int_{I_1 \times I_2} e^{2\pi i(x_1+x_2)}\,\mathrm{d}x = \int_{I_1} e^{2\pi i x_1}\,\mathrm{d}x_1 \cdot \int_{I_2} e^{2\pi i x_2}\,\mathrm{d}x_2$$

Fact: a rectangle P has an integer side length if and only if $\phi(P) = 0$.

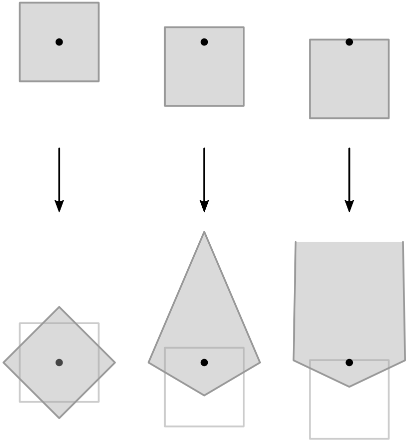
$\longrightarrow$ Stan Wagon, *"Fourteen Proofs of a Result About Tiling a Rectangle"*

Dual volumes and the canonical form

Ω

Polar duality

(polar) dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq 1 \text{ for all } y \in P\}$.



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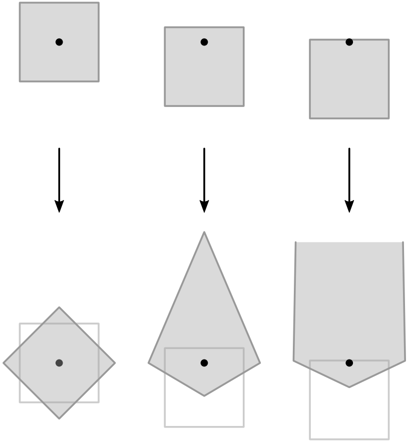

Central new idea: the volume of the dual behaves valuative!

Dual volumes

canonical form...

$$\Omega(P;x) := \mathrm{vol}(P-x)^\circ = \frac{p(x)}{q(x)}$$

Observe: this is a rational function in x.

$\Longrightarrow$ Ω can be extended to points x outside of P.

Theorem. (Arkani-Hamed, Bai, Lam; 2017)

$$\Omega(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n; x) = \Omega(P_1;x) + \cdots + \Omega(P_n;x).$$

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$$\Omega(P;x) \cdot \prod_F L_F(x) = r(x)$$

- $L_F(x) := h_F - \langle u_F, x\rangle$ ... facet-defining linear form
- u_F ... unit normal vector
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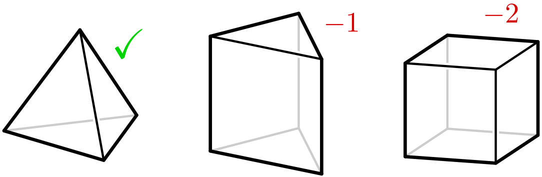

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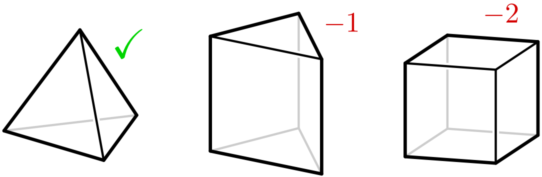

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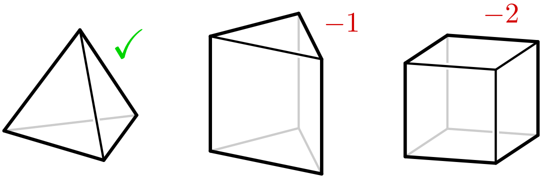

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Example: for the d-cube $\square_d := [-1,1]^d$ we have

$$\Omega(\square_d; x) = \frac{\text{some constant}}{\prod_i (1 - x_i^2)} \quad \Longrightarrow \quad \mathrm{drop}(\square_d) = d - 1.$$

The drop under composition

Lemma.

$$\mathrm{drop}(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) \geq \min_i \mathrm{drop}(P_i).$$

Proof. Observe

$$\deg \Omega(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \deg \Big(\sum_i \Omega(P_i) \Big) \leq \max_i \deg \Omega(P_i).$$

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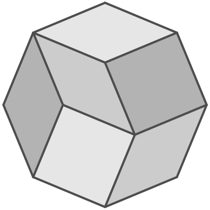

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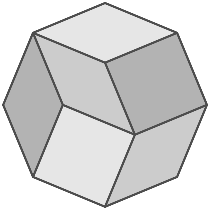
Questions:

- What other polytopes have a drop?
- What characterizes polytopes with a particular drop s?



Properties of the drop

(i) $\mathrm{drop}(P_1 \times \cdots \times P_n) = n - 1 + \sum_i \mathrm{drop}(P_i)$.

(ii) if F is a facet of P, then

$$\mathrm{drop}(F) \geq \mathrm{drop}(P) - 1,$$

with equality if and only if P has a facet F' parallel to F.

(iii) $\mathrm{drop}(P) \leq d - 1$.

(iv) $\mathrm{drop}(SP + t) = \mathrm{drop}(P)$.

(v) if π is a projection onto a hyperplane, then

$$\mathrm{drop}(\pi P) \geq \mathrm{drop}(P) - 1.$$

(vi) $\mathrm{drop}(P_1 + \cdots + P_n) \geq (d - 1) - \sum_i (d_i - 1) + \sum_i \mathrm{drop}(P_i)$.

(vii) if P is centrally symmetric

$$\mathrm{drop}(P) \text{ is } \begin{cases} \text{even} & \text{if } d \text{ is odd} \\ \text{odd} & \text{if } d \text{ is even} \end{cases}.$$

Maximal drop

Lemma.

A zonotope $P \subset \mathbb{R}^d$ *attains the maximal possible* $\mathrm{drop}(P) = d - 1$.

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Proof. (actually, four proofs) We have $\mathrm{drop}(P) \le d - 1$, but also a zonotope ...

1. ... is a projection of an n-cube $\square_n$:

$$\mathrm{drop}(\pi_d \square_n) \ge \underbrace{\mathrm{drop}(\square_n)}_{=n-1} - (n - d) = d - 1.$$

2. ... is a Minkowski sum of line segments $S_1, ..., S_n$:

$$\mathrm{drop}(S_1 + \cdots + S_n) \ge (d-1) - \sum_i \underbrace{(\dim(S_i) - 1)}_{=0} + \sum_i \underbrace{\mathrm{drop}(S_i)}_{=0} = d - 1.$$

3. ... can be tiled by parallelepipeds $P_1, ..., P_n$:

$$\mathrm{drop}(P_1 \uplus \cdots \uplus P_n) \ge \min_i \underbrace{\mathrm{drop}(P_i)}_{=d-1} = d - 1.$$

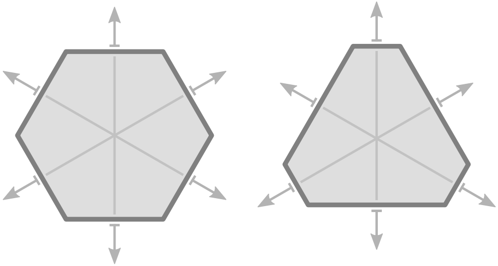
4. ... 2-faces are centrally symmetric:

$$\mathrm{drop}(P) \ge \underbrace{\mathrm{drop}(\text{2-face})}_{\in\{0,1\}} + (d - 2) = d - 1.$$

□

What else has a drop?

Observation: for maximal drop facets must come in parallel pairs.



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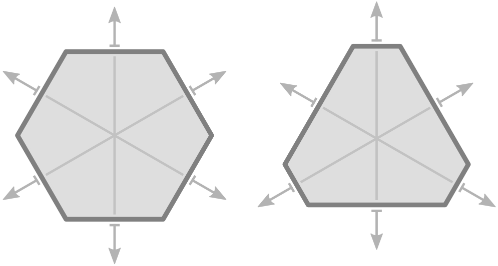

Question: can a non-centrally symmetric polygon have a drop?

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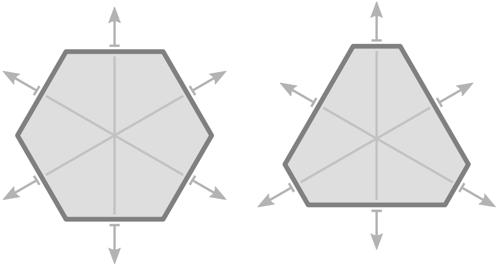
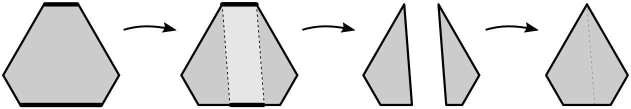
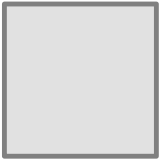
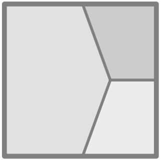


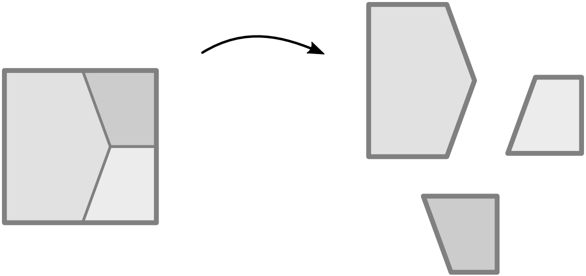
Question: can a non-centrally symmetric polygon have a drop?

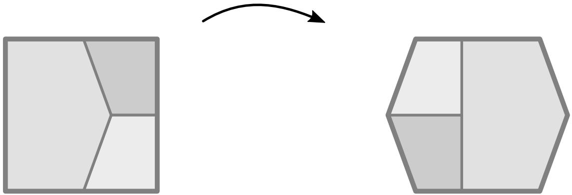
"Proof" that the answer is No:

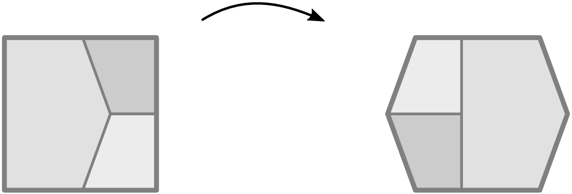












$$
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&= \phi(P_1 + t_1) + \cdots + \phi(P_n + t_n) \\
&= \phi\big((P_1 + t_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (P_n + t_n)\big) = \phi(Q)
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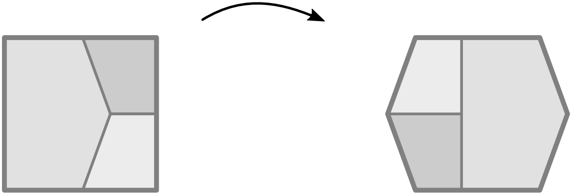

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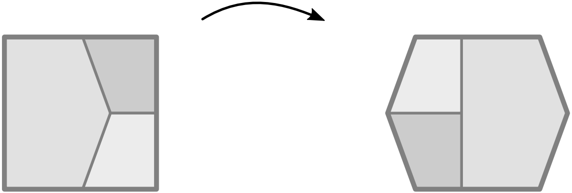
Translation scissors congruence



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A new translation-invariant valuation

$$\Omega_0$$

The view from infinity

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{(-1)^m \prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting Ω to the hyperplane at infinite (given by $x_0 = 0$).
- restricting the numerator (resp. denominator) to the "expected leading monomials".

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Lemma.

Ω_0 *is a translation-invariant valuation.* (but Ω is not)

Proof idea. Translations preserve the leading coefficients of a polynomial:

$$p(x) = \sum_\alpha p_\alpha x^\alpha \quad\longrightarrow\quad p(x+t) = \sum_\alpha p_\alpha (x+t)^\alpha. \qquad \square$$

How to use Ω_0

Observation: $\Omega_0(P) = 0$ if and only if $\operatorname{drop}(P) > 0$.

Theorem.

If P and Q are translation scissors congruent, then

$$\operatorname{drop}(P) > 0 \quad \Longleftrightarrow \quad \operatorname{drop}(Q) > 0.$$

But ...

- We can only distinguish drop vs. no-drop.
- We lose all information about the precise value of the degree drop.

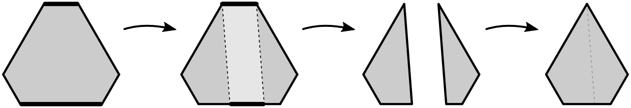
CENTRAL SYMMETRY $\Leftrightarrow$ drop $= 1$

Theorem.

For $d = 2$ we have $\mathrm{drop}(P) > 0$ if and only if P is centrally-symmetric.

Proof.

- every edge needs a parallel edge $\implies$ must be a $2n$-gon



- $\Omega_0(P) = 0$ and this is preserved in all steps $\lightning$ $\square$

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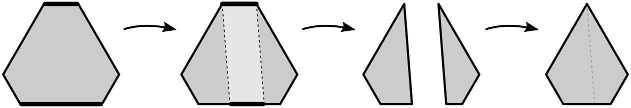

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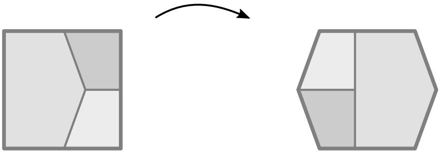

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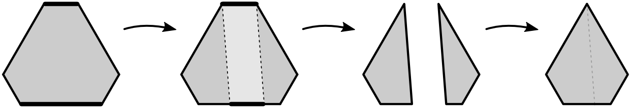

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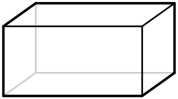
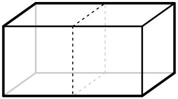
Theorem.

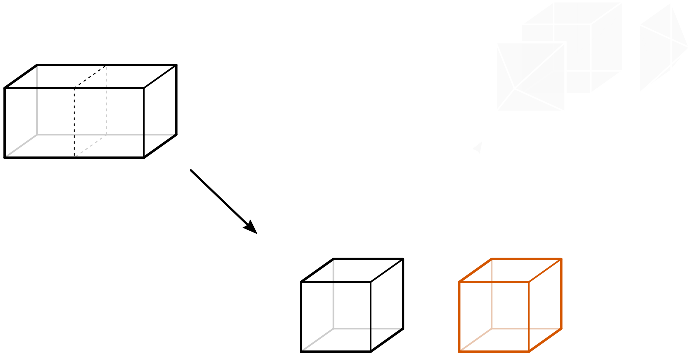
P has maximal degree drop $d - 1$ if and only if P is a zonotope.

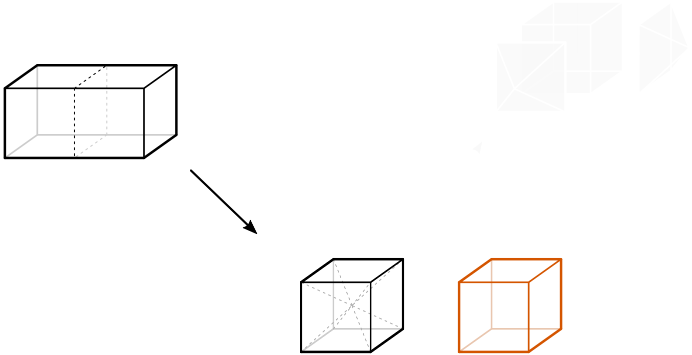
Proof.

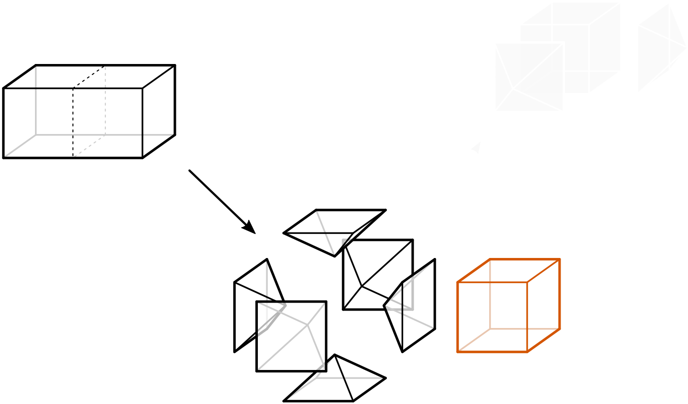
- if P has maximal drop, then so do its faces.
- all 2-faces centrally symmetric $\implies$ zonotope. □

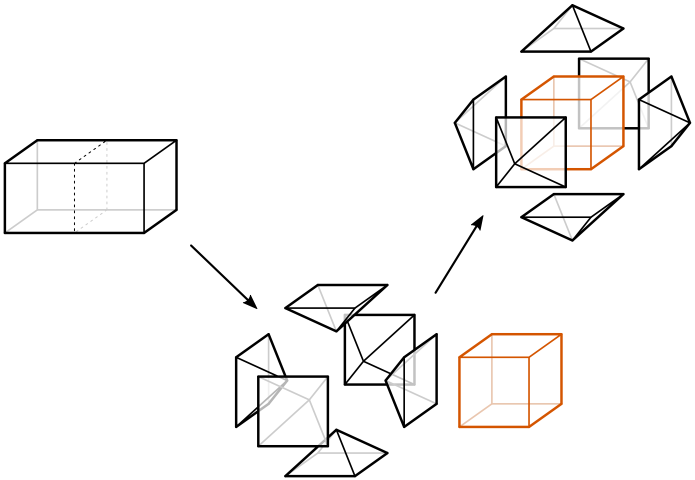


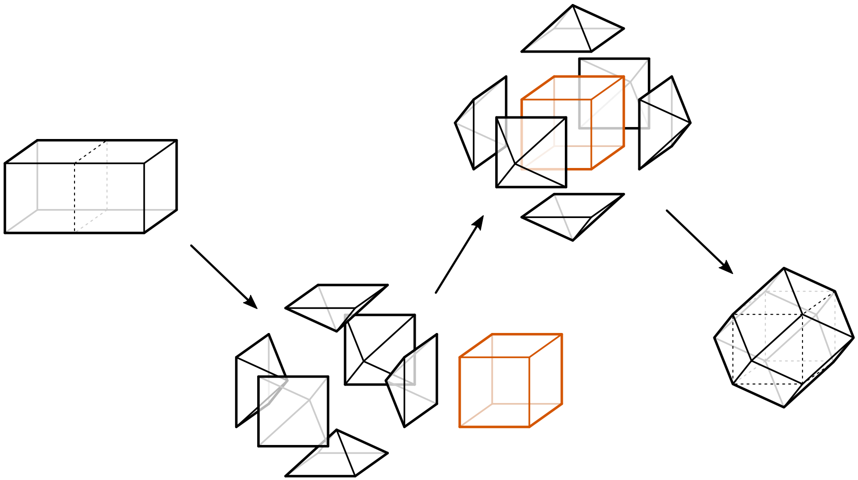


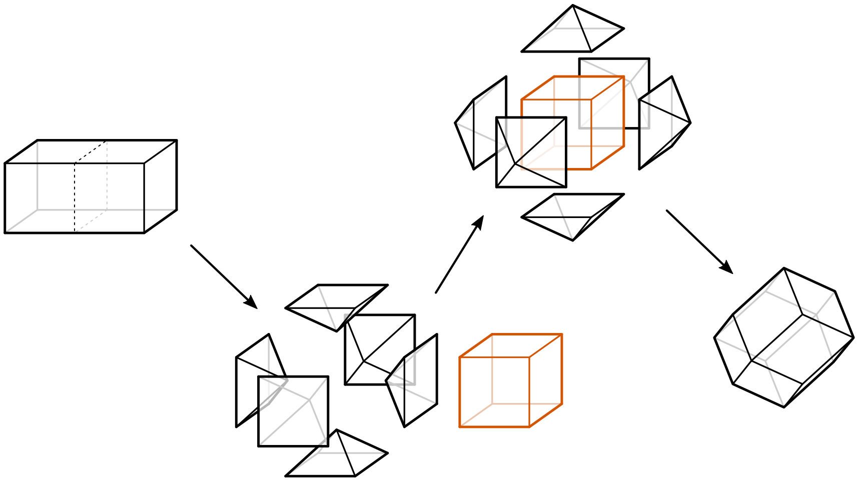


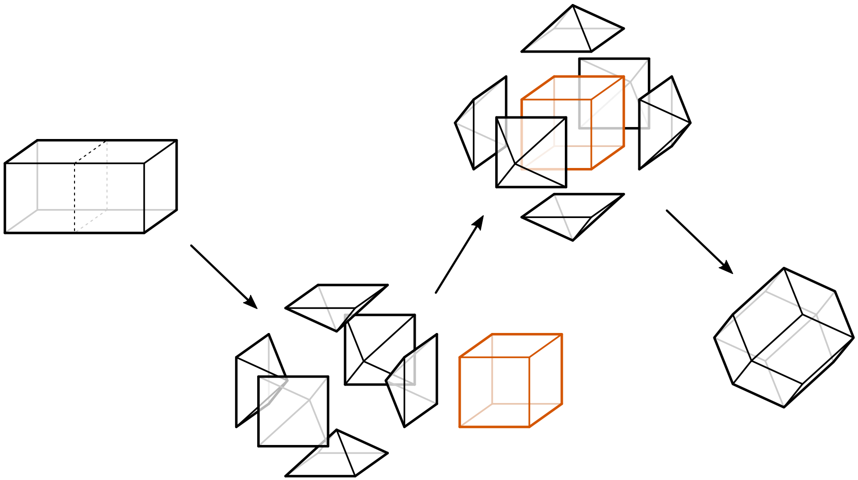












Question: Are zonotopes only translation scissors congruent to zonotopes?
or stronger, is the precise degree drop preserved under TS congruence?

Yes and no

Theorem.

In dimension $d \leq 3$ the degree drop is a translation scissors invariant.

Proof. (for $d = 3$)

$$\mathrm{drop}(P) = \begin{cases} 0 & \Omega_0 \neq 0 \\ 1 & \Omega_0 = 0 \text{ and } P \text{ is not centrally symmetric} \\ 2 & \Omega_0 = 0 \text{ and } P \text{ is centrally symmetric} \end{cases}.$$

Both $\Omega_0 = 0$ and being centrally symmetric are TS invariant. □

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In dimension $d \leq 3$, being a zonotope is a translation scissors invariant.

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Corollary.

In dimension $d \leq 3$, being a zonotope is a translation scissors invariant.

This is not true in dimensions $d \geq 4$.

Example: 4-cube and 24-cell.

Homogeneity

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A valuation is ***k*-homogeneous** if for all $\lambda > 0$ holds

$$\phi(\lambda P) = \lambda^k \phi(P).$$

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Lemma.

Ω_0 is 1-homogeneous. *(but Ω is not)*

Proof.

$$\begin{aligned}\Omega(\lambda P; x) &= \mathrm{vol}(\lambda P - x)^\circ \\ &= \mathrm{vol}(\lambda(P - x/\lambda))^\circ \\ &= \mathrm{vol}(\lambda^{-1}(P - x/\lambda)^\circ) \\ &= \lambda^{-d}\,\mathrm{vol}(P - x/\lambda)^\circ = \lambda^{-d}\,\Omega(P; x/\lambda).\end{aligned}$$

$$\begin{aligned}\Omega_0(\lambda P; x) = \lambda^{-d}\,\Omega(P; 0, x/\lambda) &= \lambda^{-d}\frac{\mathrm{adj}_P(0, x/\lambda)}{\prod_F L_F(0, x/\lambda)} \\ &= \lambda^{-d}\frac{\lambda^{-(m-d-1)}\,\mathrm{adj}_P(0,x)}{\lambda^{-m}\prod_F L_F(0,x)} = \lambda\frac{\mathrm{adj}_P(0,x)}{\prod_F L_F(0,x)} = \lambda\,\Omega_0(P;x).\end{aligned}$$

□

Homogeneity is great!

Theorem. (McMullen)

If Ω_0 is 1-homogeneous, then it is **Minkowski additive***:*

$$\Omega_0(P_1 + \cdots + P_n) = \Omega_0(P_1) + \cdots + \Omega_0(P_n).$$

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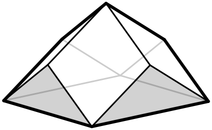
Observation: Minkowski sums of low-dimensional polytopes have a degree drop.



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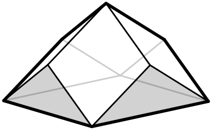

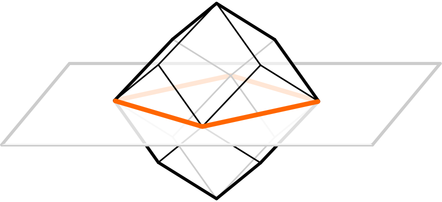

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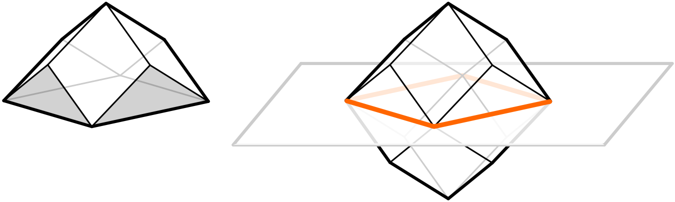

Theorem.

If P is a centrally-symmetric polytope of odd dimension with $\mathrm{drop}(P) > 0$, then each half Q of a central dissection has $\mathrm{drop}(Q) > 0$ as well.

A characterization in dimension three

Theorem.

If P is a 3-dimensional polytope, then

$$\mathrm{drop}(P) = \begin{cases} 0 & \textit{if } P + (-P) \textit{ is } \underline{\textit{not}} \textit{ a zonotope} \\ 1 & \textit{if } P + (-P) \textit{ is a zonotope, but } P \textit{ itself is } \underline{\textit{not}} \\ 2 & \textit{if } P \textit{ is a zonotope} \end{cases}.$$

We currently have no such characterization in higher dimensions.

McMullen's decomposition

Theorem. (McMullen)

If Ω_0 is translation-invariant, 1-homogeneous and weakly continuous, then there is a valuation ϕ on $(d-1)$-dimensional cones so that

$$\Omega_0(P) = \sum_{e\subset P} \ell_e \, \phi(N_P(e)).$$

Questions:

- How to verify weak continuity?
- How to determine the valuation ϕ?

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Theorem.

$$\Omega_0(P;x) = -\frac{1}{\|x\|^2} \sum_{e \subset P} \ell_e \, \Omega\big(T_P(e)\big).$$

McMullen's decomposition for $d = 2$

Theorem.

For $d = 2$ holds

$$\Omega_0(P; x) = -\frac{1}{\|x\|^2} \sum_{e \subset P} \frac{\ell_e}{\langle x, u_e \rangle}.$$

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Case study: the triangle

$$\frac{-\operatorname{adj}_\Delta}{\langle x, u_1\rangle\langle x, u_2\rangle\langle x, u_3\rangle} = -\frac{1}{\|x\|^2}\Big(\frac{\ell_1}{\langle x, u_1\rangle} + \frac{\ell_2}{\langle x, u_2\rangle} + \frac{\ell_3}{\langle x, u_3\rangle}\Big)$$

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$$\operatorname{adj}_\Delta = \frac{\operatorname{Area}(\Delta)}{\operatorname{CircR}(\Delta)}.$$

McMullen's decomposition for simplices

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$$\Omega_0(P;x) = -\frac{1}{\|x\|^2}\sum_e \ell_e\,\Omega\big(T_P(e);x\big).$$

First proof idea: triangulate P + prove theorem for simplices.

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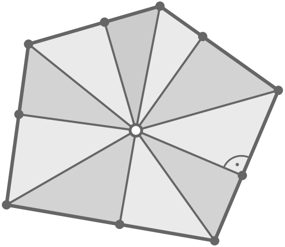
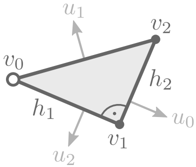
$$\overbrace{\det\begin{pmatrix} | & | & & | \\ u_0 & u_1 & \cdots & u_d \\ | & | & & | \\ h_0 & h_1 & \cdots & h_d \end{pmatrix}}^{\mathrm{adj}_\Delta = \Omega_0(P;x)\cdot\prod_F\langle x,u_F\rangle}\|x\|^2$$

$$= \sum_{i<j}(-1)^{i+j+d}\underbrace{\det\begin{pmatrix} | & & | & & | & & | \\ u_0 & \cdots & v_i & \cdots & v_j & \cdots & u_d \\ | & & | & & | & & | \\ 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix}}_{\ell_{ij}\,\Omega(T_P(e_{ij}))\cdot\prod_F\langle x,u_F\rangle}\langle u_i,x\rangle\langle u_j,x\rangle.$$

Second proof idea: orthoschemes





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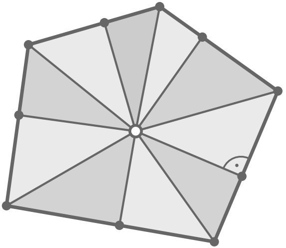
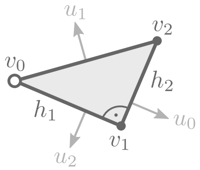
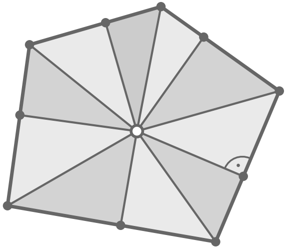
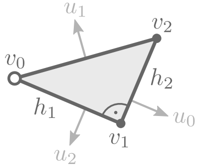




$$
\begin{aligned}
v_0 &= (0, 0, 0, ..., 0), & u_0 &= (h_0, 0, 0, ..., 0, 0), \\
v_1 &= (h_1, 0, 0, ..., 0), & u_1 &= (-h_2, h_1, 0, ..., 0, 0), \\
v_2 &= (h_1, h_2, 0, ..., 0), & u_2 &= (0, -h_3, h_2, ..., 0, 0), \\
v_3 &= (h_1, h_2, h_3, ..., 0), & u_3 &= (0, 0, -h_4, h_3, ..., 0, 0), \\
&\vdots & &\vdots \\
v_d &= (h_1, h_2, h_3, ..., h_d), & u_d &= (0, 0, 0, ..., 0, -h_{d+1}),
\end{aligned}
$$

Second proof idea: orthoschemes





$$\sum_{i=1}^{d} x_i^2 = -\sum_{\substack{i,j=0 \\ i<j}}^{d} \frac{h_{i+1}^2 + \cdots + h_j^2}{h_i h_{i+1} h_j h_{j+1}} (h_{i+1}x_i - h_i x_{i+1})(h_{j+1}x_j - h_j x_{j+1}).$$

Thank you.