Adjoint degrees and scissors congruence for polytopes

Martin Winter
(joint work with Tom Baumbach, Ansgar Freyer and Julian Weigert)

TU Berlin + MPI

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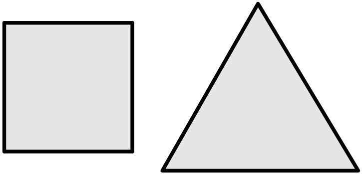
Scissors congruence

Two polytopes P and Q are **scissors congruent** if

$$P = P_1 \,\dot\cup\, \cdots \,\dot\cup\, P_n \qquad Q = Q_1 \,\dot\cup\, \cdots \,\dot\cup\, Q_n.$$

with $Q_i = S_i(P_i)$, where $S_i \in \mathrm{Iso}(\mathbb{R}^d)$ are isometries.



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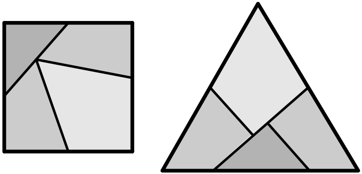

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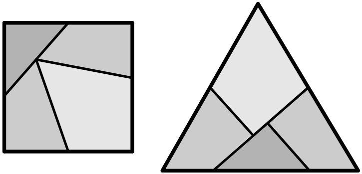

Theorem (Wallace, Bolyai, Gerwien; 1807/33/35)

Two polygons P, Q are scissors congruent if and only if $\mathrm{vol}(P) = \mathrm{vol}(Q)$.

Hilbert's Third Problem

Given any two polyhedra P and Q of equal volume, is it always possible to dissect P into finitely many polyhedral pieces $P_1, ..., P_n$, which can then be reassembled to yield Q?

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Theorem. (Dehn; 1901)

If $P, Q \subset \mathbb{R}^3$ are scissors congruent, then they have the same Dehn invariant.

$$D(P) := \sum_{e \subset P} \mathrm{len}(e) \otimes_{\mathbb{Z}} \theta(e)/2\pi \;\in\; \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}.$$

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Theorem. (Sydler; 1965)

$P, Q \subset \mathbb{R}^3$ are scissors congruent if and only if they have the same volume and same Dehn invariant.

Valuations

Whenever P, Q, $P \cap Q$ and $P \cup Q$ are polytopes, a **valuation** satisfies

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$$

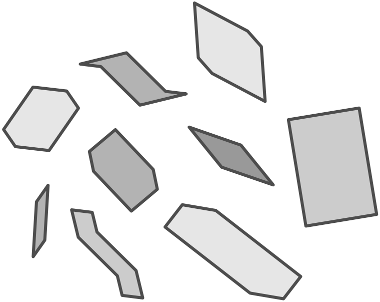
... but what we actually care about:

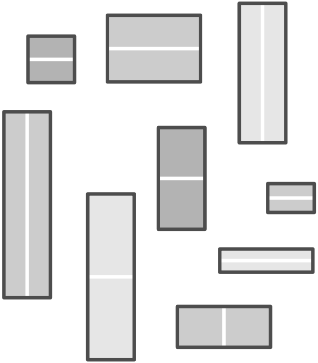
$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

Examples:

- volume
- surface area measure
- Euler characteristic
- mixed volumes
- number of contained lattice points
- ...

Two composition puzzles





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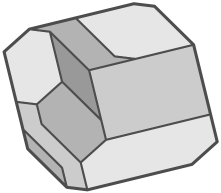

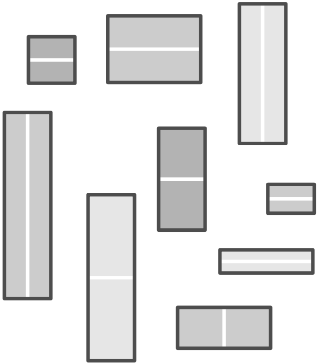

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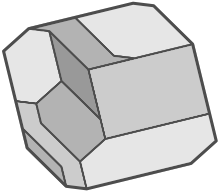
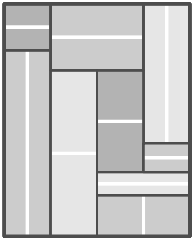
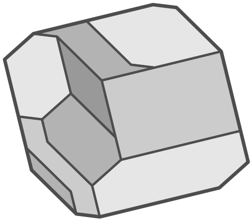




Puzzle I

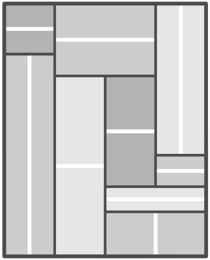


Let $\nu(P)$ be the *surface area measure* of $P \subset \mathbb{R}^d$ on $\mathbb{S}^{d-1}$.

$$\phi(P) := \nu(P) - \nu(-P)$$

Observe: a polygon P is centrally symmetric if and only if $\phi(P) = 0$.

Puzzle II



$$\phi(P) := \int_P e^{2\pi i(x_1+x_2)}\,\mathrm{d}x = \int_P e^{2\pi i x_1}\,\mathrm{d}x \cdot \int_P e^{2\pi i x_2}\,\mathrm{d}x$$

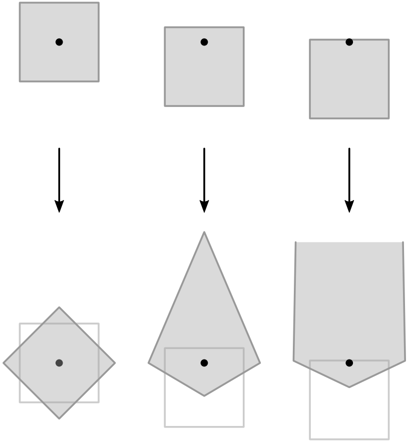
Observe: a rectangle P has an integer side length if and only if $\phi(P) = 0$.

Dual volumes and the canonical form

Ω

Polar duality

(polar) dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P\}$.



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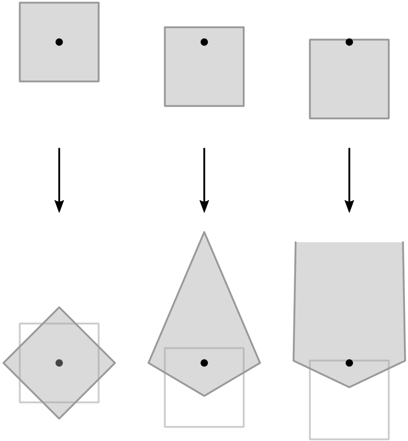

Central new idea: the volume of the dual behaves valuative!

Dual volumes

$$\Omega(P;x) := \mathrm{vol}(P-x)^\circ = \frac{p(x)}{q(x)}$$

Observe: this is a rational function in x.

$\implies$ Ω can be extended to points x outside of P.

Theorem. (Arkani-Hamed, Bai, Lam; 2017)

$$\Omega(P_1 \uplus \cdots \uplus P_n; x) = \Omega(P_1;x) + \cdots + \Omega(P_n;x).$$

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- $\ell_F(x) := h_F - \langle u_F, x\rangle$ ... facet-defining linear form
- u_F ... unit normal vector
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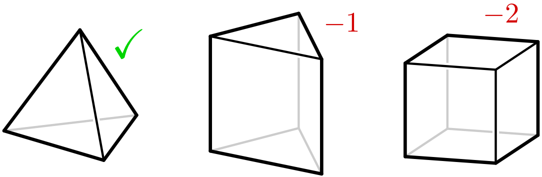

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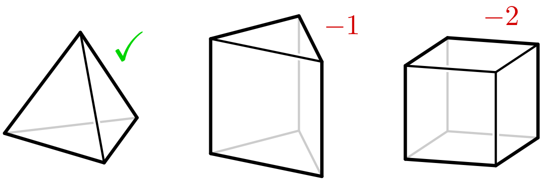

We call this defficiency in degree the **degree drop** of P:

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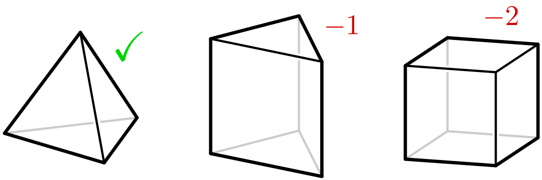

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Example: for the d-cube $\square_d := [-1,1]^d$ we have

$$\Omega(\square_d; x) = \frac{\text{some constant}}{\prod_i (1 - x_i^2)} \quad\Longrightarrow\quad \mathrm{drop}(\square_d) = d - 1.$$

Adjoint degrees under composition

Lemma.

$$\operatorname{drop}(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) \geq \min_i \operatorname{drop}(P_i).$$

Adjoint degrees under composition

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$$\mathrm{drop}(P_1 \uplus \cdots \uplus P_n) \geq \min_i \mathrm{drop}(P_i).$$

Proof.

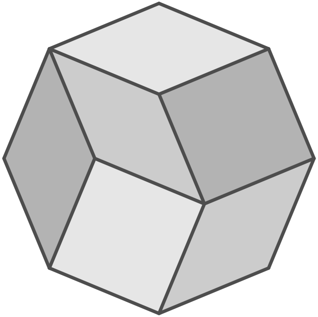
- first, homogenize:

$$\Omega(P; x_0, x) := \frac{\mathrm{adj}_P(x_0, x)}{\prod_F \ell_F(x_0, x)} \quad \begin{array}{l} \leftarrow \text{homogenized to degree } m-d-1 \\ \leftarrow \text{homogenized to degree } m \end{array}$$

- If $s := \mathrm{drop}(P_1 \uplus \cdots \uplus P_n)$ and $s_i := \mathrm{drop}(P_i)$, then

$$\begin{aligned} x_0^s \frac{p(x_0, x)}{\prod\limits_{F \subset P} \ell_F(x_0, x)} &= \Omega(P \uplus \cdots \uplus P_n; x_0, x) \\ &= \sum_i \Omega(P_i; x_0, x) = x_0^{\min s_i} \sum_i \frac{p_i(x_0, x)}{\prod\limits_{F \subset P_i} \ell_F(x_0, x)}. \end{aligned}$$

$\Longrightarrow \; s \geq \min s_i$ □



Questions:

- What characterizes the class of polytopes with drop s?
- How to tell the drop of a polytope from geometric/combinatorial characteristics?

Drop is inherited by faces

Lemma.

If $F \subset P$ is a facet, then

$$\mathrm{drop}(F) \geq \mathrm{drop}(P) - 1$$

with equality if and only of P has a facet parallel to F.

Proof.

$$\overbrace{\frac{\mathrm{adj}_F(x)}{\underbrace{\prod_{G<F} \ell_G(x)}_{m_F}}}^{m_F-(d-1)-1-s_F} = \Omega(F;x) = \overbrace{\frac{\mathrm{adj}_P(x)|_F}{\underbrace{\prod_{G\neq F} \ell_G(x)|_F}_{m-\begin{cases}2 & \text{has parallel facet}\\ 1 & \text{no parallel facet}\end{cases}}}}^{\leq m-d-1-s}$$

$$\implies s_F \geq s - 1$$

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$\implies s_F \geq s - 1$ ☐

Corollary

A d-polytope has $\mathrm{drop}(P) \leq d-1$.

Question: which polytopes have maximal degree drop?

Centrally symmetric polygons

Lemma.

A centrally symmetric polygon P *has* $\mathrm{drop}(P) = 1$. *(which is maximal)*

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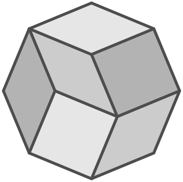
Proof I.

- a cs polygon decomposes into parallelograms $\square$



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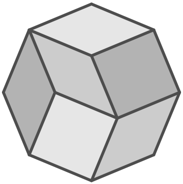

Note: zonotopes also decompose into "skew cubes" (parallelepipedes).

Lemma.

Zonotopes have maximal degree drop $d - 1$.

Centrally symmetric polygons

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A centrally symmetric polygon P has $\mathrm{drop}(P) = 1$.

Proof II.

- We have $\Omega(P;x) = \Omega(P;-x)$ due to symmetry.
- Since $\Omega = \mathrm{adj}_P \,/\, \prod_F \ell_F$, either both adj_P and $\prod_F \ell_F$ even, or both odd.
- Since P is cs, $\deg \prod_F \ell_F = m$ is even $\implies$ $\deg \mathrm{adj}_P$ is even
- Hence $\mathrm{drop}(P) = (m-2-1) - \deg \mathrm{adj}_P$ is odd $\implies$ $\mathrm{drop}(P) = 1$. □

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Note: Argument applies verbatim in higher dimensions.

Lemma.

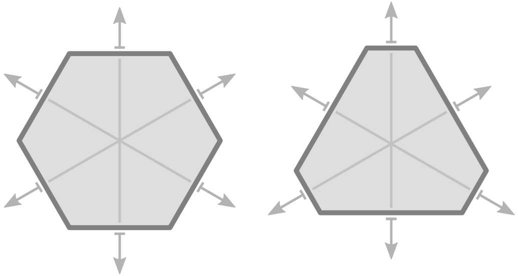
If P is centrally symmetric, then $\deg \mathrm{adj}_P$ is even, or equivalently,

$$\mathrm{drop}(P) \text{ is } \begin{cases} \text{even} & \text{if } d \text{ is odd} \\ \text{odd} & \text{if } d \text{ is even} \end{cases}$$

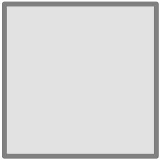
In particular, cs polytopes in even dimension have $\mathrm{drop}(P) \geq 1$.

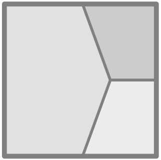
What else has a drop?

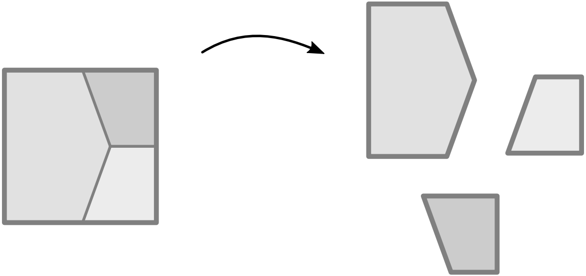
Observation: for maximal drop facets must come in parallel pairs.

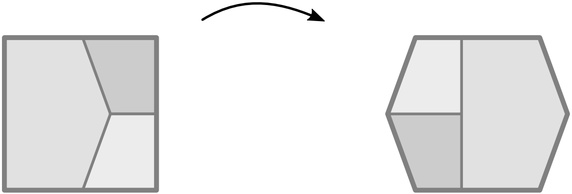


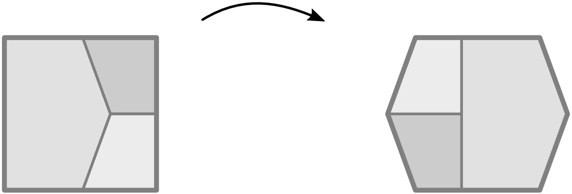
Question: can a non-centrally symmetric polygon have a drop?



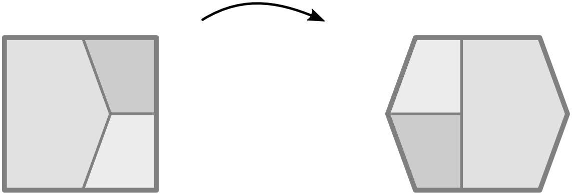








$$\begin{aligned}\phi(P) &= \phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n)\\ &= \phi(P_1) + \cdots + \phi(P_n)\\ &= \phi(P_1 + t_1) + \cdots + \phi(P_n + t_n)\\ &= \phi\big((P_1 + t_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (P_n + t_n)\big) = \phi(Q)\end{aligned}$$



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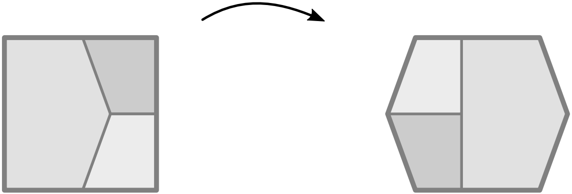
Translation scissors congruence



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A new translation-invariant valuation

$$\Omega_0$$

The view from infinity

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{\prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting Ω to the hyperplane at infinite (given by $x_0 = 0$).
- restricting the numerator (resp. denominator) to the monomials of degree $m - d - 1$ (resp. m).

The view from infinity

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{\prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting Ω to the hyperplane at infinite (given by $x_0 = 0$).
- restricting the numerator (resp. denominator) to the monomials of degree $m-d-1$ (resp. m).

Lemma.

Ω_0 *is a translation-invariant valuation.* *(but Ω is not)*

Proof idea. Translation preserve the leading coefficients of a polynomial:

$$p(x) = \sum_n p_n x^n \quad \longrightarrow \quad p(x+t) = \sum_n p_n (x+t)^n.$$

How to use Ω_0

Observation: $\Omega_0(P) = 0$ if and only if $\mathrm{drop}(P) > 0$.

Theorem.

If P and Q are translation scissors congruent, then

$$\mathrm{drop}(P) > 0 \quad \Longleftrightarrow \quad \mathrm{drop}(Q) > 0.$$

But ...

- We can only distinguish drop vs. no-drop.
- We lose all information about the precise value of the degree drop

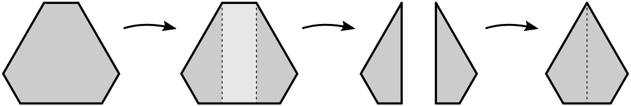
Central symmetry ⇔ drop = 1

Theorem.

For $d = 2$ we have $\mathrm{drop}(P) > 0$ if and only if P is centrally-symmetric.

Proof.

- every edge needs a parallel edge $\implies$ must be a $2n$-gon



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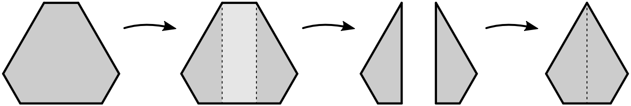

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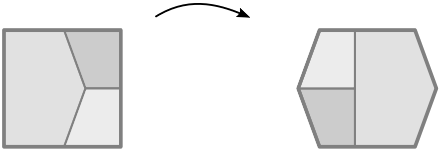

CENTRAL SYMMETRY $\Leftrightarrow$ drop $= 1$

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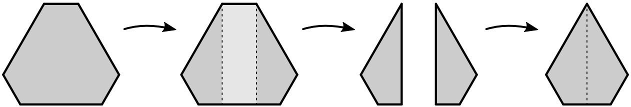


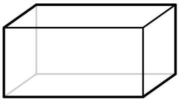
- $\Omega_0(P) = 0$ and this is preserved in all steps ↯ □

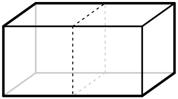
Theorem.

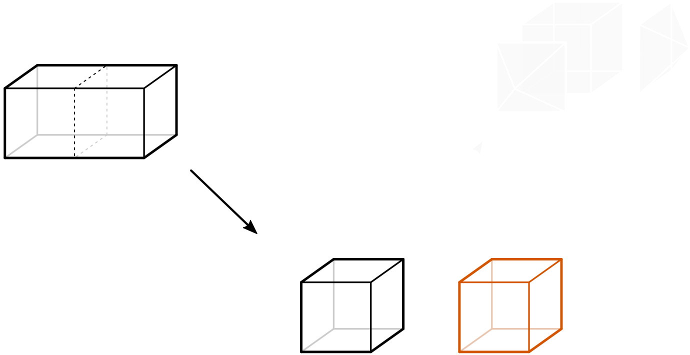
P has maximal degree drop $\mathrm{drop}(P) = d - 1$ iff P is a zonotope.

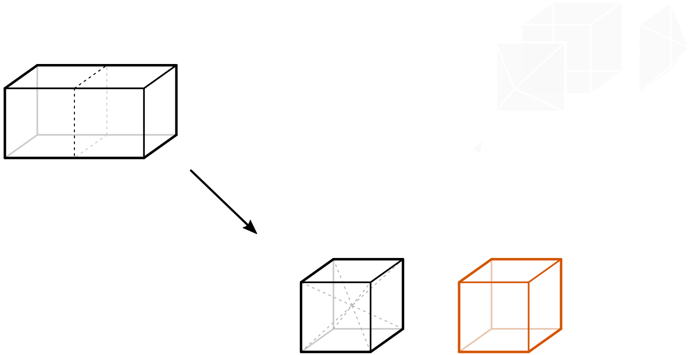
Proof.

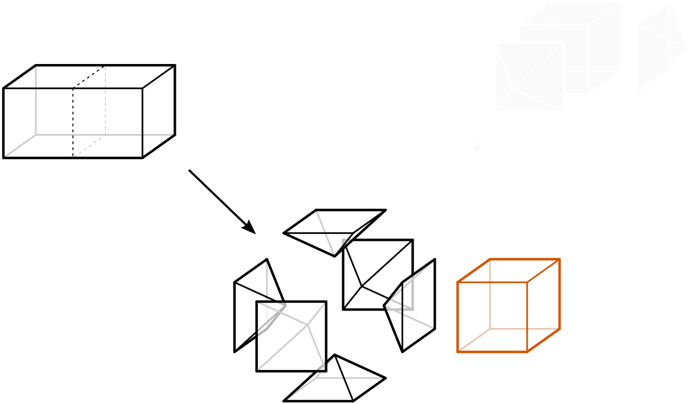
- if P has maximal drop, then so do its faces.
- all faces centrally symmetric $\implies$ zonotope. □

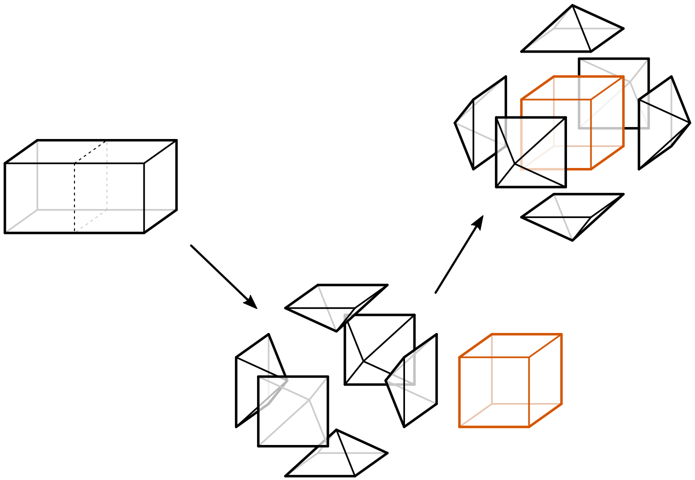


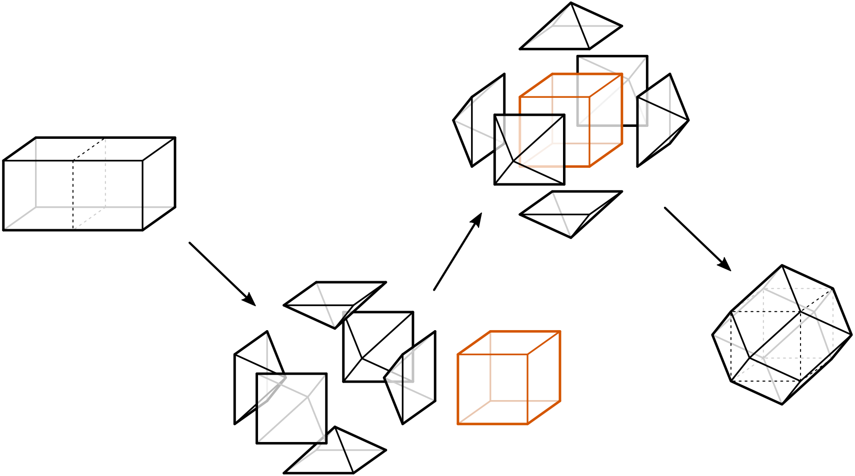


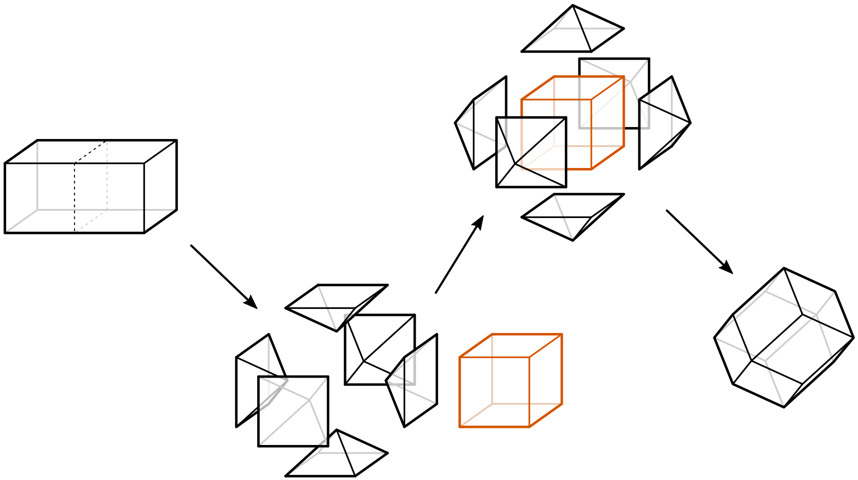


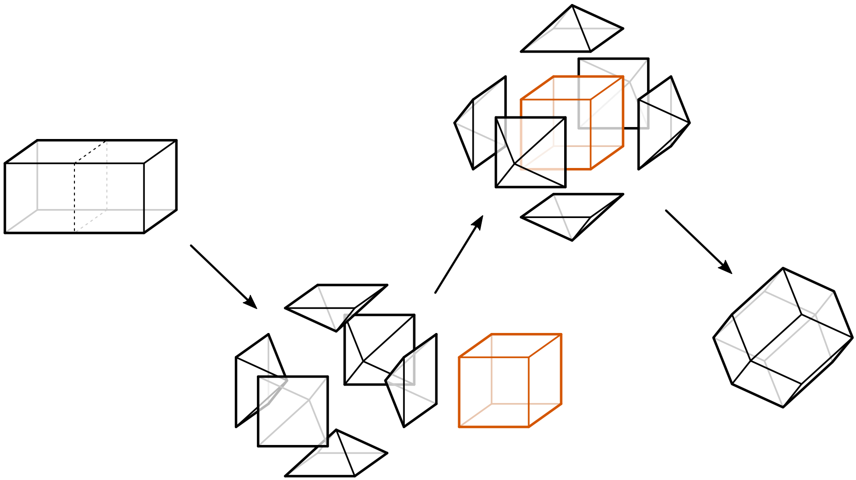












Question: Are zonotopes only translation scissors congruent to zonotopes?
or stronger, is the precise degree drop preserved under TSC?

Yes and no

Theorem.

In dimension $d \leq 3$ translation scissors congruence preserves the degree drop.

Proof. (for $d = 3$)

- if $\mathrm{drop}(P) = 0$ then $\mathrm{drop}(Q) = 0$.
- if $\mathrm{drop}(P) = 2$ then P is a zonotop, hence centrally symmetric. Both $\mathrm{drop} > 0$ and cs are preserved by TSC. But cs 3-polytopes have an even drop. Hence $\mathrm{drop}(Q) = 2$ as well.
- $\mathrm{drop}(P) = 1 \implies \mathrm{drop}(Q) = 1$ follows from $\mathrm{drop} \in \{0, 1, 2\}$. □

This is not true in dimensions $d \geq 4$.

Example: 4-cube and 24-cell.

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This is not true in dimensions $d \geq 4$.

Example: 4-cube and 24-cell.

Lemma.

In dimension $d = 3$, $\mathrm{drop}(P) > 0$ if and only if $P + (-P)$ is a zonotope.

Homogeneity

Homogeneity of Ω_0

A valuation is ***k*-homogeneous** if for $\lambda > 0$ holds

$$\phi(\lambda P) = \lambda^k \phi(P).$$

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Lemma.

Ω_0 is 1-homogeneous. *(but Ω is not)*

Proof.

$$\begin{aligned}
\Omega(\lambda P; x) &= \operatorname{vol}(\lambda P - x)^\circ \\
&= \operatorname{vol}(\lambda(P - x/\lambda))^\circ \\
&= \operatorname{vol}(\lambda^{-1}(P - x/\lambda)^\circ) \\
&= \lambda^{-d} \operatorname{vol}(P - x/\lambda)^\circ = \lambda^{-d}\, \Omega(P; x/\lambda).
\end{aligned}$$

$$\begin{aligned}
\Omega_0(\lambda P; x) &= \lambda^{-d}\, \Omega(P; 0, x/\lambda) = \lambda^{-d} \frac{\operatorname{adj}_P(0, x/\lambda)}{\prod_F \ell_F(0, x/\lambda)} \\
&= \lambda^{-d} \frac{\lambda^{-(m-d-1)} \operatorname{adj}_P(0, x)}{\lambda^{-m} \prod_F \ell_F(0, x)} = \lambda \frac{\operatorname{adj}_P(0, x)}{\prod_F \ell_F(0, x)} = \lambda\, \Omega_0(P; x).
\end{aligned}$$

□

Homogeneity is great!

Theorem. (McMullen)

If Ω_0 is 1-homogeneous, then it is **Minkowski additive***:*

$$\Omega_0(P_1 + \cdots + P_n) = \Omega_0(P_1) + \cdots + \Omega_0(P_n).$$

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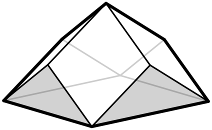
Observation: Minkowski sums of low-dimensional polytopes have a degree drop.



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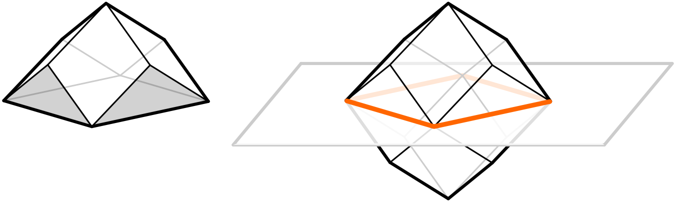

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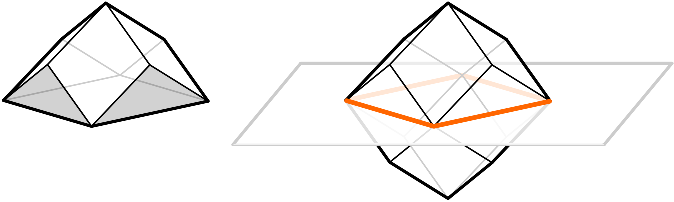

Theorem.

If P is a centrally-symmetric polytope of odd dimension with $\mathrm{drop}(P) > 0$, then each half Q of a central dissection has $\mathrm{drop}(Q) > 0$ as well.

McMullen's decomposition

Theorem. (McMullen)

If Ω_0 is translation-invariant, 1-homogeneous and weakly continuous, then there is a valuation ϕ on $(d-1)$-cones so that

$$\Omega_0(P) = \sum_{e \subset P} \operatorname{len}(e)\phi(N_P(e)).$$

Questions:

- How to verify weak continuity?
- How to determine the valuation ϕ?

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Theorem. (Baumbach, Freyer, Weigert, W.; 2025+)

$$\Omega_0(P;x) = -\frac{1}{\|x\|^2}\sum_{e} \operatorname{len}(e)\,\Omega\big(T_P(e)\big).$$

McMullen's decomposition for $d = 2$

Theorem.

For $d = 2$ holds

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Case study: the triangle

$$\frac{\mathrm{adj}_\Delta}{\langle x,u_1\rangle\langle x,u_2\rangle\langle x,u_3\rangle} = -\frac{1}{\|x\|^2}\Big(\frac{\ell_1}{\langle x,u_1\rangle} + \frac{\ell_2}{\langle x,u_2\rangle} + \frac{\ell_3}{\langle x,u_3\rangle}\Big)$$

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$$= \frac{\mathrm{Area}(\Delta)}{\mathrm{CircR}(\Delta)}.$$

Composition of Ω_0

Theorem. (Baumbach, Freyer, Weigert, W.; 2025+)

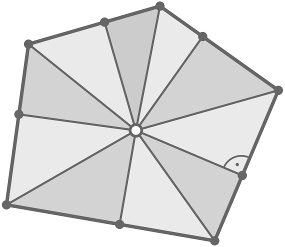
$$\Omega_0(P;x) = -\frac{1}{\|x\|^2} \sum_e \operatorname{len}(e)\, \Omega\big(T_P(e)\big).$$

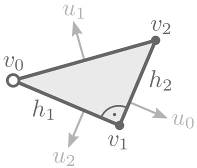
First proof idea: triangulate P + proof theorem for simplices.

$$\overbrace{\det \begin{pmatrix} | & | & & | \\ u_0 & u_1 & \cdots & u_d \\ | & | & & | \\ h_0 & h_1 & \cdots & h_d \end{pmatrix}}^{\operatorname{adj}_\Delta = \Omega_0(P;x)\cdot \prod_F \langle u_F, x\rangle} \|x\|^2$$

$$= -\sum_{i<j} \underbrace{\det \begin{pmatrix} | & & | & & | & & | \\ u_0 & \cdots & v_i & \cdots & v_j & \cdots & u_d \\ | & & | & & | & & | \\ 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix}}_{\operatorname{len}(e_{ij})\, \Omega(T_P(e_{ij}))\cdot \prod_F \langle u_F, x\rangle} \langle u_i, x\rangle \langle u_j, x\rangle.$$

Second proof idea: orthoschemes





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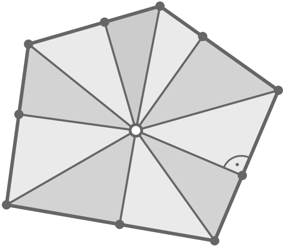
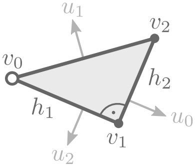
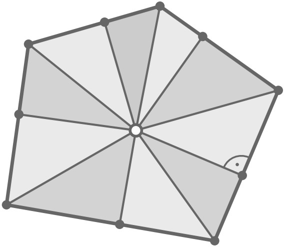


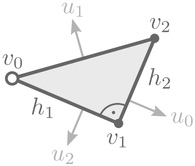


$$v_0 = (0, 0, 0, ..., 0),$$
$$v_1 = (h_1, 0, 0, ..., 0),$$
$$v_2 = (h_1, h_2, 0, ..., 0),$$
$$v_3 = (h_1, h_2, h_3, ..., 0),$$
$$\vdots$$
$$v_d = (h_1, h_2, h_3, ..., h_d),$$

$$u_0 = (h_0, 0, 0, ..., 0, 0),$$
$$u_1 = (-h_2, h_1, 0, ..., 0, 0),$$
$$u_2 = (0, -h_3, h_2, ..., 0, 0),$$
$$u_3 = (0, 0, -h_4, h_3, ..., 0, 0),$$
$$\vdots$$
$$u_d = (0, 0, 0, ..., 0, -h_{d+1}),$$

Second proof idea: orthoschemes





$$\sum_{i=1}^{d} x_i^2 = -\sum_{\substack{i,j=0 \\ i<j}}^{d} \frac{h_{i+1}^2 + \cdots + h_j^2}{h_i h_{i+1} h_j h_{j+1}} (h_{i+1} x_i - h_i x_{i+1})(h_{j+1} x_j - h_j x_{j+1}).$$

Thank you.