# Adjoint degrees and scissors congruence for polytopes

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combinatorial synergies

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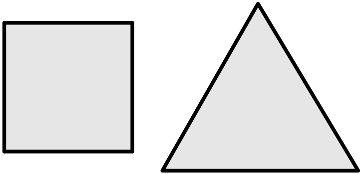
# Scissors congruence

Two polytopes $P$ and $Q$ are **scissors congruent** if

$$P = P_1 \,\dot\cup\, \cdots \,\dot\cup\, P_n \qquad Q = Q_1 \,\dot\cup\, \cdots \,\dot\cup\, Q_n.$$

with $Q_i = S_i(P_i)$, where $S_i \in \mathrm{Iso}(\mathbb{R}^d)$ are isometries.



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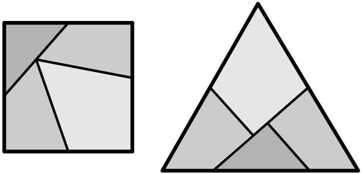

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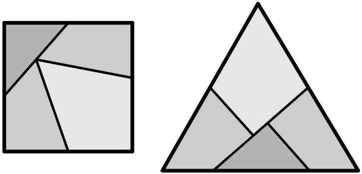

**Theorem** (Wallace, Bolyai, Gerwien; 1807/33/35)

*Two polygons $P, Q$ are scissors congruent if and only if $\mathrm{vol}(P) = \mathrm{vol}(Q)$.*

# Hilbert's Third Problem

> *Given any two polyhedra $P$ and $Q$ of equal volume, is it always possible to dissect $P$ into finitely many polyhedral pieces $P_1, ..., P_n$, which can then be reassembled to yield $Q$?*
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**Theorem.** (Dehn; 1901)

*If $P, Q \subset \mathbb{R}^3$ are scissors congruent, then they have the same Dehn invariant.*

$$D(P) := \sum_{e \subset P} \operatorname{len}(e) \otimes_{\mathbb{Z}} \theta(e)/2\pi \;\in\; \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}.$$

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**Theorem.** (Sydler; 1965)

*$P, Q \subset \mathbb{R}^3$ are scissors congruent if and only if they have the same volume and same Dehn invariant.*

# Valuations

Whenever $P$, $Q$, $P \cap Q$ and $P \cup Q$ are polytopes, a **valuation** satisfies

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$$

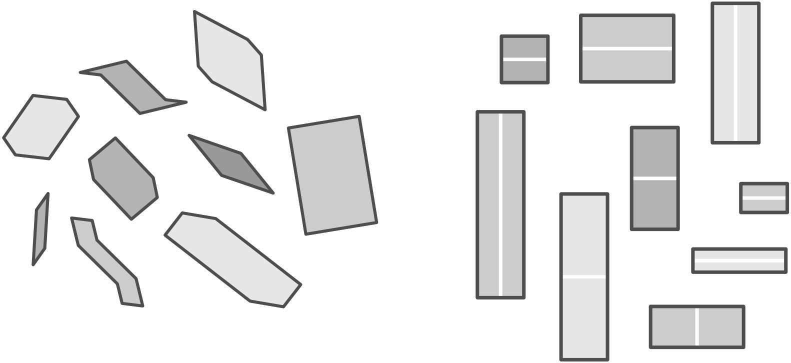
... but what we actually care about:

$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

**Examples:**

- volume
- Dehn invariant
- surface area measure
- Euler characteristic
- mixed volumes
- number of contained lattice points
- ...

# Two composition puzzles



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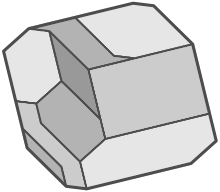

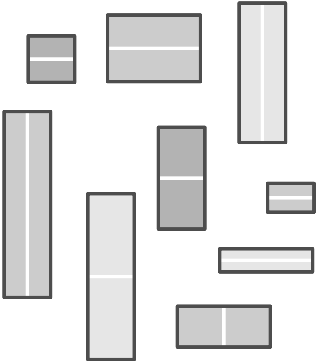

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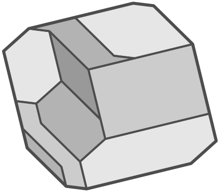
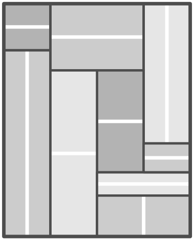
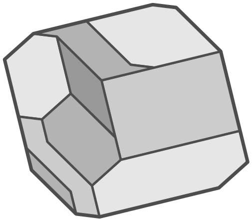




# Puzzle I

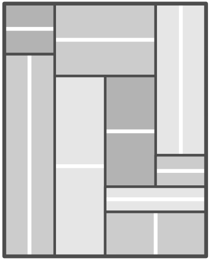


Let $\nu(P)$ be the *surface area measure* of $P \subset \mathbb{R}^d$ on $\mathbb{S}^{d-1}$.

$$\phi(P) := \nu(P) - \nu(-P)$$

**Observe:** a polygon $P$ is centrally symmetric if and only if $\phi(P) = 0$.

# Puzzle II



$$\phi(P) := \int_P e^{2\pi i(x_1+x_2)}\,\mathrm{d}x = \int_P e^{2\pi i x_1}\,\mathrm{d}x \cdot \int_P e^{2\pi i x_2}\,\mathrm{d}x$$

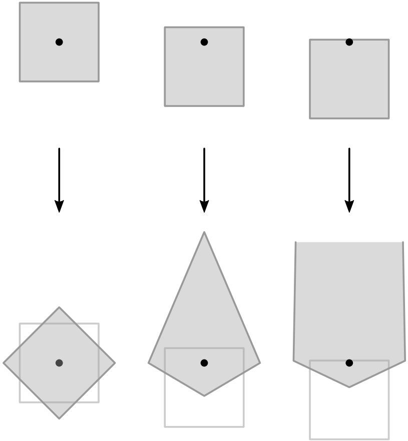
**Observe:** a rectangle $P$ has an integer side length if and only if $\phi(P) = 0$.

# Dual volumes and the canonical form

$$\Omega$$

# Polar duality

(polar) dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq 1 \text{ for all } y \in P\}.$



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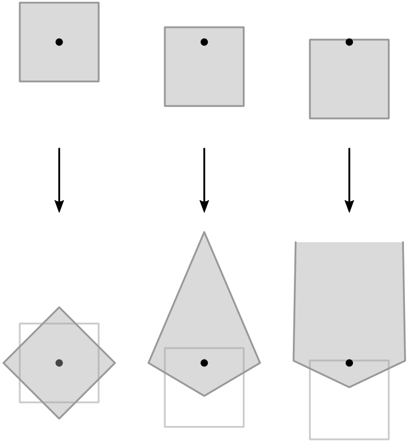

**Central new idea:** the volume of the dual behaves valuative!

# Dual volumes

$$\Omega(P;x) := \mathrm{vol}(P-x)^\circ = \frac{p(x)}{q(x)}$$

**Observe:** this is a rational function in $x$.

$\implies$ $\Omega$ can be extended to points $x$ outside of $P$.

**Theorem.** (Arkani-Hamed, Bai, Lam; 2017)

$$\Omega(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n; x) = \Omega(P_1;x) + \cdots + \Omega(P_n;x).$$

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$$\Omega(P;x) \cdot \prod_F \ell_F(x) = r(x)$$

- $\ell_F(x) := h_F - \langle u_F, x\rangle$ ... facet-defining linear form
- $u_F$ ... unit normal vector
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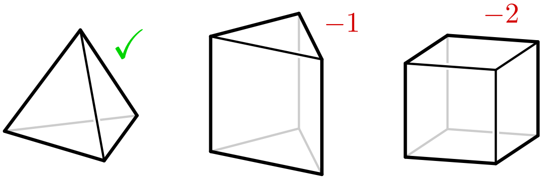

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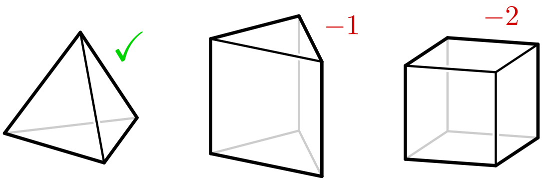


We call this defficiency in degree the **degree drop** of $P$:

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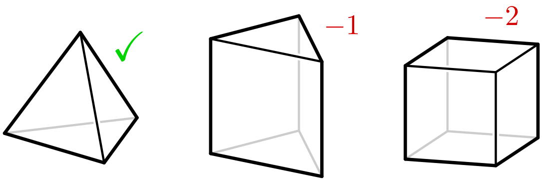


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**Example:** for the $d$-cube $\square_d := [-1,1]^d$ we have

$$\Omega(\square_d; x) = \frac{\text{some constant}}{\prod_i (1 - x_i^2)} \quad\Longrightarrow\quad \mathrm{drop}(\square_d) = d - 1.$$

# Adjoint degrees under composition

**Lemma.**

$$\mathrm{drop}(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) \geq \min_i \mathrm{drop}(P_i).$$

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$$\operatorname{drop}(P_1 \uplus \cdots \uplus P_n) \geq \min_i \operatorname{drop}(P_i).$$

*Proof.*

- first, homogenize:

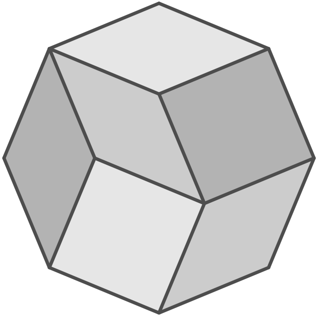
$$\Omega(P; x_0, x) := \frac{\operatorname{adj}_P(x_0, x)}{\prod_F \ell_F(x_0, x)} \quad \begin{matrix} \leftarrow \text{homogenized to degree } m-d-1 \\ \leftarrow \text{homogenized to degree } m \end{matrix}$$

- If $s := \operatorname{drop}(P_1 \uplus \cdots \uplus P_n)$ and $s_i := \operatorname{drop}(P_i)$, then

$$\begin{aligned} x_0^s \frac{p(x_0, x)}{\prod\limits_{F \subset P} \ell_F(x_0, x)} &= \Omega(P \uplus \cdots \uplus P_n; x_0, x) \\ &= \sum_i \Omega(P_i; x_0, x) \;=\; x_0^{\min s_i} \sum_i \frac{p_i(x_0, x)}{\prod\limits_{F \subset P_i} \ell_F(x_0, x)}. \end{aligned}$$

$\implies \; s \geq \min s_i$ $\square$

# Adjoint degrees under composition



**Questions:**

- What other polytopes have a drop?
- What characterizes polytopes with a particular drop $s$?

# Drop is inherited by faces

**Lemma.**

*If $F \subset P$ is a facet, then*

$$\mathrm{drop}(F) \geq \mathrm{drop}(P) - 1$$

*with equality if and only of $P$ has a facet parallel to $F$.*

*Proof.*

$$\frac{\overset{m_F-(d-1)-1-s_F}{\mathrm{adj}_F(x)}}{\underset{m_F}{\prod_{G<F} \ell_G(x)}} = \Omega(F;x) = \frac{\overset{\leq m-d-1-s}{\mathrm{adj}_P(x)|_F}}{\underset{m - \begin{cases} 2 & \text{has parallel facet} \\ 1 & \text{no parallel facet} \end{cases}}{\prod_{G \neq F} \ell_G(x)|_F}}$$

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$$\implies s_F \geq s-1$$

□

**Corollary**

*A $d$-polytope has* $\operatorname{drop}(P) \leq d-1$.

**Question:** which polytopes have maximal degree drop?

# Centrally symmetric polygons

**Lemma.**

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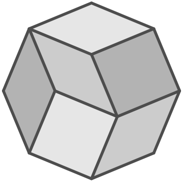

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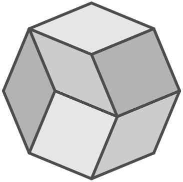

**Note:** zonotopes also decompose into "skew cubes" (parallelepipedes).

**Lemma.**

*Zonotopes have maximal degree drop $d - 1$.*

# Centrally symmetric polygons

**Lemma.**

*A centrally symmetric polygon $P$ has* $\mathrm{drop}(P) = 1$.

*Proof II.*

- We have $\Omega(P;x) = \Omega(P;-x)$ due to symmetry.
- Since $\Omega = \mathrm{adj}_P \,/ \prod_F \ell_F$, either both $\mathrm{adj}_P$ and $\prod_F \ell_F$ even, or both odd.
- Since $P$ is cs, $\deg \prod_F \ell_F = m$ is even $\implies$ $\deg \mathrm{adj}_P$ is even
- Hence $\mathrm{drop}(P) = (m-2-1) - \deg \mathrm{adj}_P$ is odd $\implies$ $\mathrm{drop}(P) = 1$. $\square$

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**Note:** Argument applies verbatim in higher dimensions.

**Lemma.**

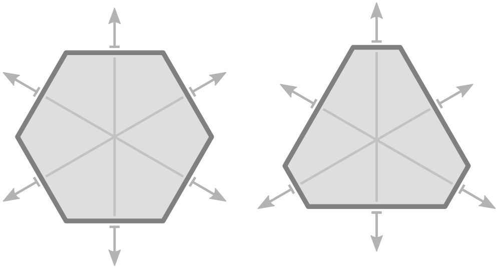
*If $P$ is centrally symmetric, then $\deg \mathrm{adj}_P$ is even, or equivalently,*

$$\mathrm{drop}(P) \text{ is } \begin{cases} \text{even} & \text{if } d \text{ is odd} \\ \text{odd} & \text{if } d \text{ is even} \end{cases}$$

*In particular, cs polytopes in even dimension have $\mathrm{drop}(P) \geq 1$.*

# What else has a drop?

**Observation:** for maximal drop facets must come in parallel pairs.



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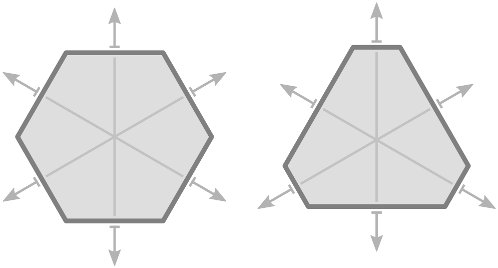

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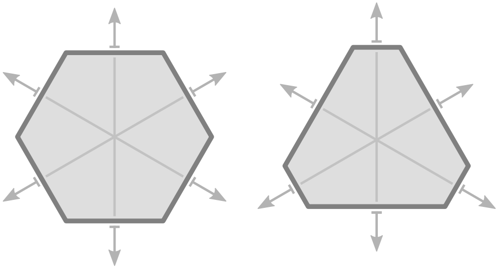
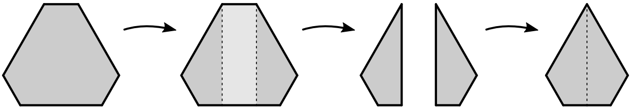
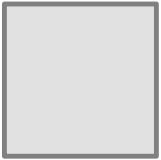
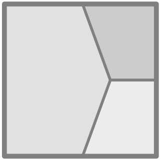


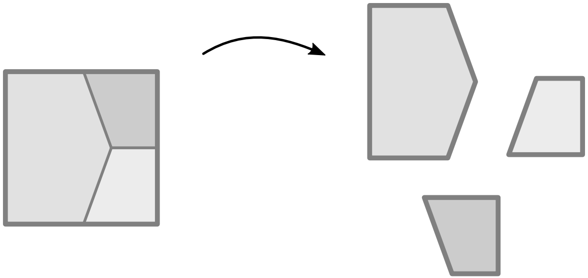
**Question:** can a non-centrally symmetric polygon have a drop?

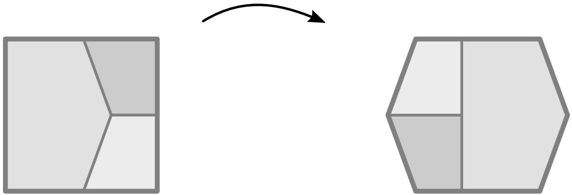
*"Proof" that the answer is No:*

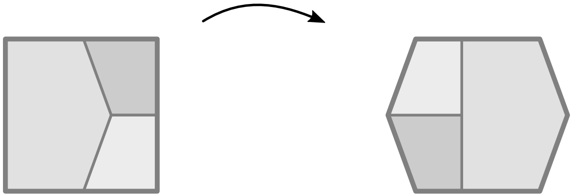




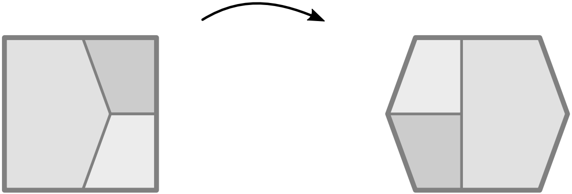








$$
\begin{aligned}
\phi(P) &= \phi(P_1 \cupdot \cdots \cupdot P_n) \\
&= \phi(P_1) + \cdots + \phi(P_n) \\
&= \phi(P_1 + t_1) + \cdots + \phi(P_n + t_n) \\
&= \phi\big((P_1 + t_1) \cupdot \cdots \cupdot (P_n + t_n)\big) = \phi(Q)
\end{aligned}
$$



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&\overset{?}{=} \phi(P_1 + t_1) + \cdots + \phi(P_n + t_n) \\
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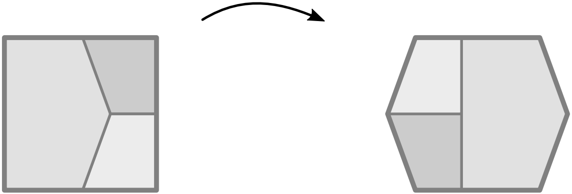
$$

# Translation scissors congruence



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# A new translation-invariant valuation

$$\Omega_0$$

# The view from infinity

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{\prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting $\Omega$ to the hyperplane at infinite (given by $x_0 = 0$).
- restricting the numerator (resp. denominator) to the monomials of degree $m-d-1$ (resp. $m$).

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**Lemma.**

$\Omega_0$ *is a translation-invariant valuation.* *(but $\Omega$ is not)*

*Proof idea.* Translation preserve the leading coefficients of a polynomial:

$$p(x) = \sum_n p_n x^n \quad\longrightarrow\quad p(x+t) = \sum_n p_n (x+t)^n.$$

# How to use $\Omega_0$

**Observation:** $\Omega_0(P) = 0$ if and only if $\mathrm{drop}(P) > 0$.

**Theorem.**

*If $P$ and $Q$ are translation scissors congruent, then*

$$\mathrm{drop}(P) > 0 \quad \Longleftrightarrow \quad \mathrm{drop}(Q) > 0.$$

**But ...**

- We can only distinguish drop vs. no-drop.
- We lose all information about the precise value of the degree drop

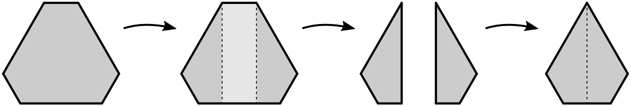
# CENTRAL SYMMETRY $\Leftrightarrow$ drop $= 1$

**Theorem.**

*For $d = 2$ we have $\mathrm{drop}(P) > 0$ if and only if $P$ is centrally-symmetric.*

*Proof.*

- every edge needs a parallel edge $\implies$ must be a $2n$-gon



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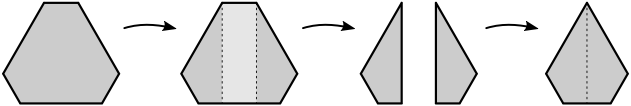

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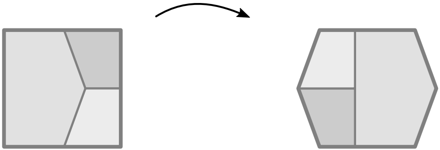

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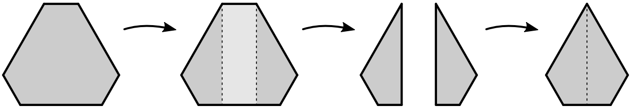


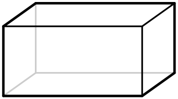
- $\Omega_0(P) = 0$ and this is preserved in all steps ↯ $\square$

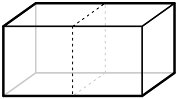
**Theorem.**

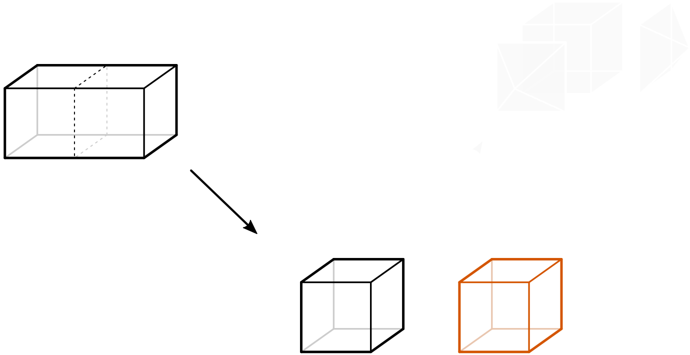
*$P$ has maximal degree drop $\mathrm{drop}(P) = d - 1$ iff $P$ is a zonotope.*

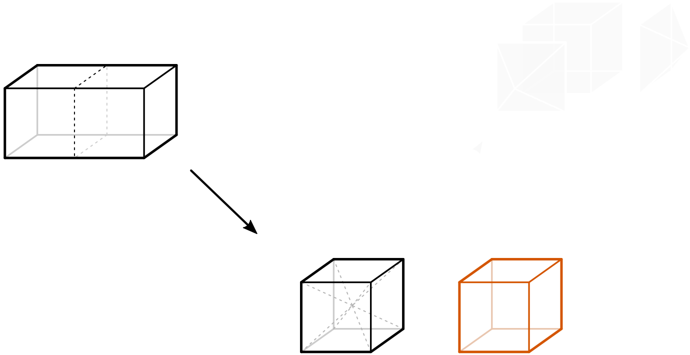
*Proof.*

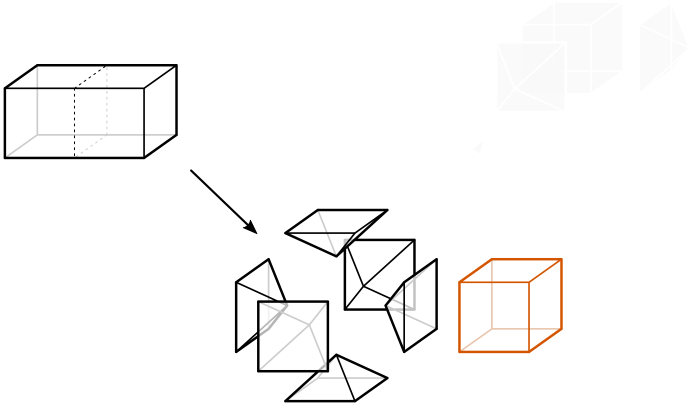
- if $P$ has maximal drop, then so do its faces.
- all faces centrally symmetric $\implies$ zonotope. $\square$

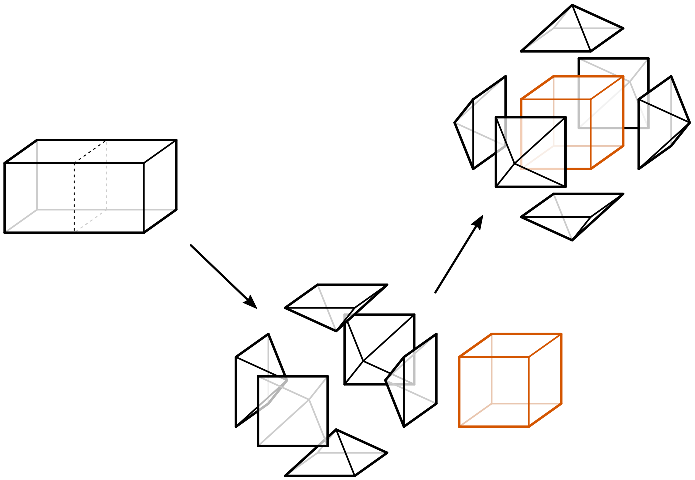


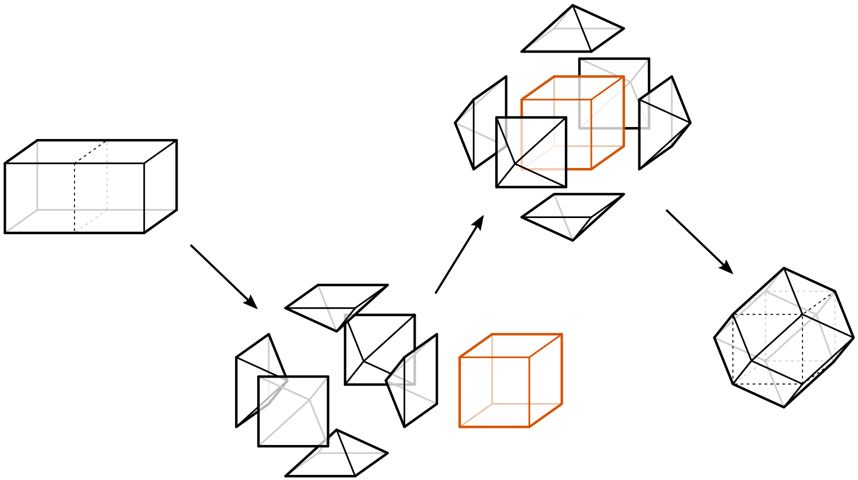


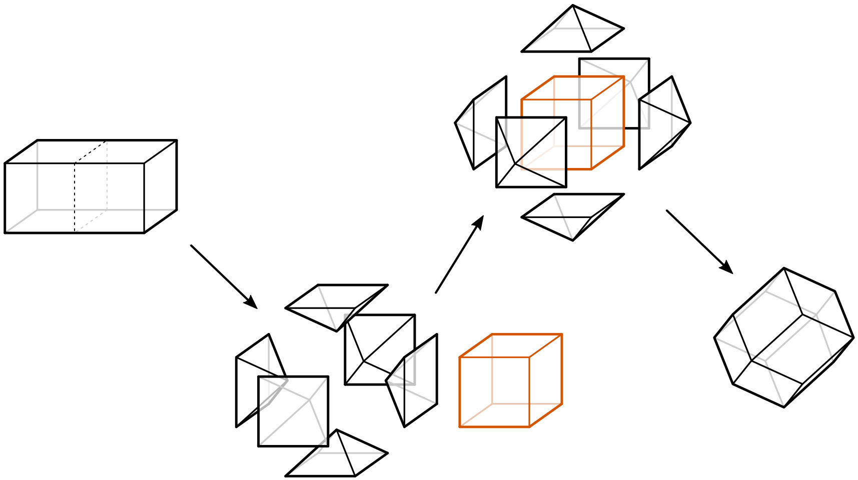


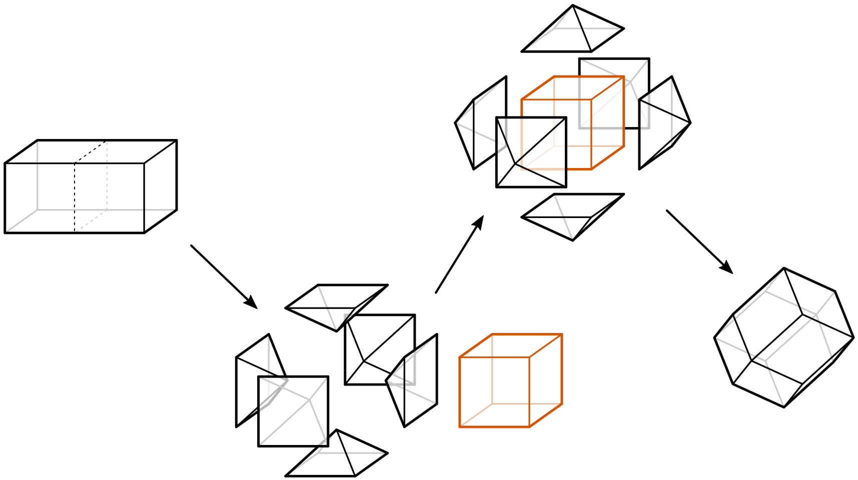












**Question:** Are zonotopes only translation scissors congruent to zonotopes?
or stronger, is the precise degree drop preserved under TSC?

# Yes and no

**Theorem.**

*In dimension $d \leq 3$ translation scissors congruence preserves the degree drop.*

*Proof.* (for $d = 3$)

- if $\mathrm{drop}(P) = 0$ then $\mathrm{drop}(Q) = 0$.
- if $\mathrm{drop}(P) = 2$ then $P$ is a zonotop, hence centrally symmetric. Both $\mathrm{drop} > 0$ and cs are preserved by TSC. But cs 3-polytopes have an even drop. Hence $\mathrm{drop}(Q) = 2$ as well.
- $\mathrm{drop}(P) = 1 \implies \mathrm{drop}(Q) = 1$ follows from $\mathrm{drop} \in \{0, 1, 2\}$. □

This is not true in dimensions $d \geq 4$.

**Example:** 4-cube and 24-cell.

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This is not true in dimensions $d \geq 4$.

**Example:** 4-cube and 24-cell.

**Lemma.**

*In dimension $d = 3$, $\mathrm{drop}(P) > 0$ if and only if $P + (-P)$ is a zonotope.*

# Homogeneity

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A valuation is ***k*-homogeneous** if for $\lambda > 0$ holds

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*$\Omega_0$ is 1-homogeneous.* *(but $\Omega$ is not)*

*Proof.*

$$\begin{aligned}\Omega(\lambda P; x) &= \mathrm{vol}(\lambda P - x)^\circ \\ &= \mathrm{vol}(\lambda(P - x/\lambda))^\circ \\ &= \mathrm{vol}(\lambda^{-1}(P - x/\lambda)^\circ) \\ &= \lambda^{-d}\,\mathrm{vol}(P - x/\lambda)^\circ = \lambda^{-d}\,\Omega(P; x/\lambda).\end{aligned}$$

$$\begin{aligned}\Omega_0(\lambda P; x) &= \lambda^{-d}\,\Omega(P; 0, x/\lambda) = \lambda^{-d} \frac{\mathrm{adj}_P(0, x/\lambda)}{\prod_F \ell_F(0, x/\lambda)} \\ &= \lambda^{-d} \frac{\lambda^{-(m-d-1)}\,\mathrm{adj}_P(0,x)}{\lambda^{-m} \prod_F \ell_F(0,x)} = \lambda \frac{\mathrm{adj}_P(0,x)}{\prod_F \ell_F(0,x)} = \lambda\,\Omega_0(P; x).\end{aligned}$$

$\square$

# Homogeneity is great!

**Theorem.** (McMullen)

*If $\Omega_0$ is 1-homogeneous, then it is* **Minkowski additive***:*

$$\Omega_0(P_1 + \cdots + P_n) = \Omega_0(P_1) + \cdots + \Omega_0(P_n).$$

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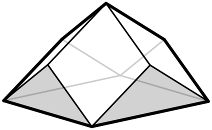
**Observation:** Minkowski sums of low-dimensional polytopes have a degree drop.



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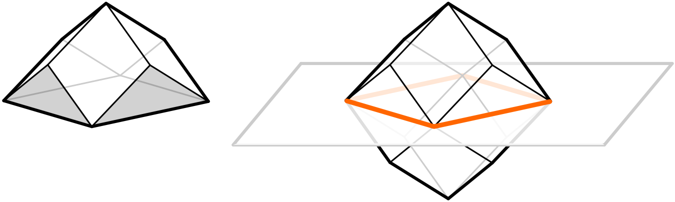

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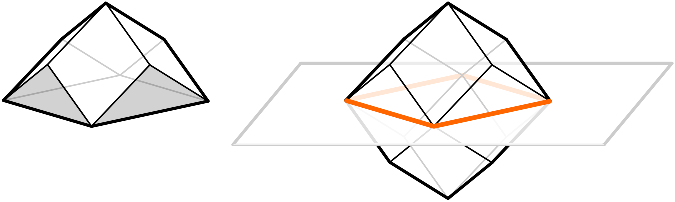

**Theorem.**

*If $P$ is a centrally-symmetric polytope of odd dimension with $\operatorname{drop}(P) > 0$, then each half $Q$ of a central dissection has $\operatorname{drop}(Q) > 0$ as well.*

# McMullen's decomposition

**Theorem.** (McMullen)

*If $\Omega_0$ is translation-invariant, 1-homogeneous and weakly continuous, then there is a valuation $\phi$ on $(d-1)$-cones so that*

$$\Omega_0(P) = \sum_{e \subset P} \operatorname{len}(e)\phi(N_P(e)).$$

**Questions:**

- How to verify weak continuity?
- How to determine the valuation $\phi$?

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**Theorem.** (Baumbach, Freyer, Weigert, W.; 2025+)

$$\Omega_0(P;x) = -\frac{1}{\|x\|^2} \sum_{e} \operatorname{len}(e)\,\Omega\big(T_P(e)\big).$$

# McMullen's decomposition for $d = 2$

**Theorem.**

*For $d = 2$ holds*

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$$\frac{\mathrm{adj}_\Delta}{\langle x,u_1\rangle\langle x,u_2\rangle\langle x,u_3\rangle} = -\frac{1}{\|x\|^2}\Big(\frac{\ell_1}{\langle x,u_1\rangle}+\frac{\ell_2}{\langle x,u_2\rangle}+\frac{\ell_3}{\langle x,u_3\rangle}\Big)$$

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$$= \frac{\mathrm{Area}(\Delta)}{\mathrm{CircR}(\Delta)}.$$

# Composition of $\Omega_0$

**Theorem.** (Baumbach, Freyer, Weigert, W.; 2025+)

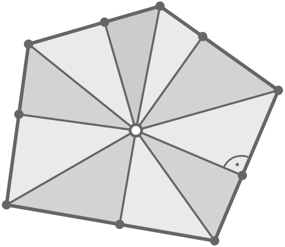
$$\Omega_0(P;x) = -\frac{1}{\|x\|^2}\sum_e \mathrm{len}(e)\,\Omega\big(T_P(e)\big).$$

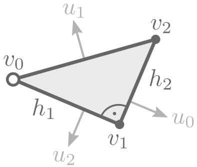
*First proof idea:* triangulate $P$ + proof theorem for simplices.

$$\overbrace{\det\begin{pmatrix} | & | & & | \\ u_0 & u_1 & \cdots & u_d \\ | & | & & | \\ h_0 & h_1 & \cdots & h_d \end{pmatrix}}^{\mathrm{adj}_\triangle = \Omega_0(P;x)\cdot\prod_F\langle u_F, x\rangle}\|x\|^2$$

$$= -\sum_{i<j}\underbrace{\det\begin{pmatrix} | & & | & & | & & | \\ u_0 & \cdots & v_i & \cdots & v_j & \cdots & u_d \\ | & & | & & | & & | \\ 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix}}_{\mathrm{len}(e_{ij})\,\Omega(T_P(e_{ij}))\cdot\prod_F\langle u_F, x\rangle}\langle u_i, x\rangle\langle u_j, x\rangle.$$

# Second proof idea: orthoschemes





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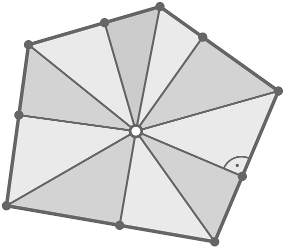

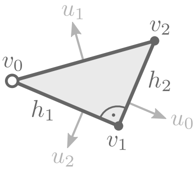


$$
\begin{aligned}
v_0 &= (0, 0, 0, ..., 0), \\
v_1 &= (h_1, 0, 0, ..., 0), \\
v_2 &= (h_1, h_2, 0, ..., 0), \\
v_3 &= (h_1, h_2, h_3, ..., 0), \\
&\vdots \\
v_d &= (h_1, h_2, h_3, ..., h_d),
\end{aligned}
$$

$$
\begin{aligned}
u_0 &= (h_0, 0, 0, ..., 0, 0), \\
u_1 &= (-h_2, h_1, 0, ..., 0, 0), \\
u_2 &= (0, -h_3, h_2, ..., 0, 0), \\
u_3 &= (0, 0, -h_4, h_3, ..., 0, 0), \\
&\vdots \\
u_d &= (0, 0, 0, ..., 0, -h_{d+1}),
\end{aligned}
$$

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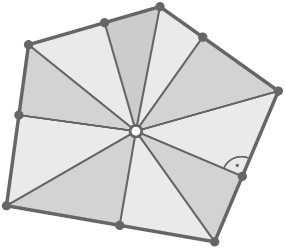

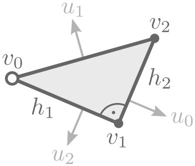


$$\sum_{i=1}^{d} x_i^2 = -\sum_{\substack{i,j=0\\ i<j}}^{d} \frac{h_{i+1}^2 + \cdots + h_j^2}{h_i h_{i+1} h_j h_{j+1}} (h_{i+1}x_i - h_i x_{i+1})(h_{j+1}x_j - h_j x_{j+1}).$$

# Thank you.