Two-level polytopes and the conjectures of Mahler and Kalai

Martin Winter
(joint work with Raman Sanyal, Jan Stricker and Matthias Schymura)

TU Berlin / MPI Leipzig

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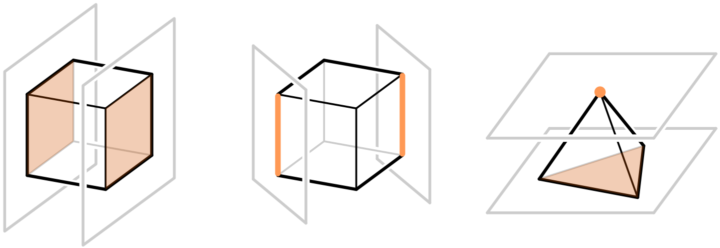
2-LEVEL POLYTOPES

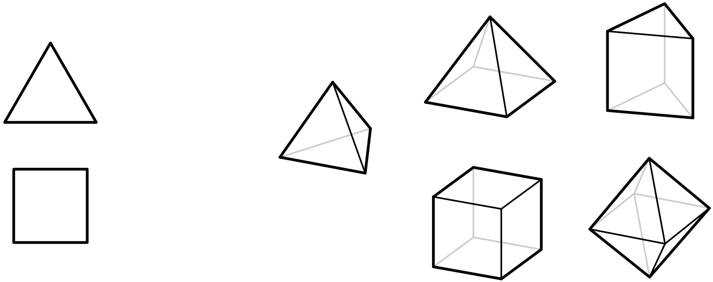
$$P = \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d,\ d \geq 0$$

Definition.

- Two faces $F_1, F_2 \subseteq P$ are **antipodal** if they are contained in parallel hyperplanes (*i.e.* there are parallel hyperplanes $H_1, H_2 \subseteq \mathbb{R}^d$ with $F_i = P \cap H_i$)
- A polytope P is **2-level** if each facet is contained in an antipodal face pair that covers all vertices.



Examples



dim	0	1	2	3	4	5	6	7	8
2-level	1	1	2	5	19	106	1150	27291	1378453

Examples

Many 2-level polytopes are constructed from combinatorial objects:

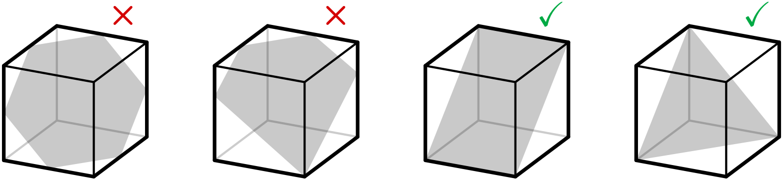
- *Hanner polytopes* (in relation to cographs)
- order polytopes of posets
- stable set polytopes of perfect graphs
  $+$ their twisted prisms ($=$ *Hansen polytopes*)
- spanning tree polytopes of series-parallel graphs
- Birkhoff polytopes (from double stochastic matrices)
- certain matroid base polytopes

Properties

- all faces are 2-level
- closed under products and joins
- #vertices $\cdot$ #facets $\leq d2^{d+1}$ (Kupavskii, Weltge; 2020)
- are 01-polytopes (if P is d-dimensional then $P \subseteq [0,1]^d$)

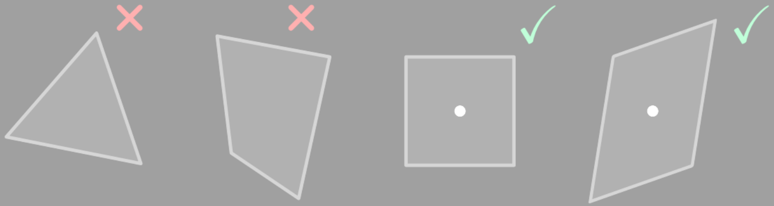
Theorem.

2-level polytopes are precisely the polytopes that can be written as the intersection of a cube with an affine subspace that is spanned by vertices of the cube.



Conjectures for centrally symmetric polytopes

$$P = -P$$



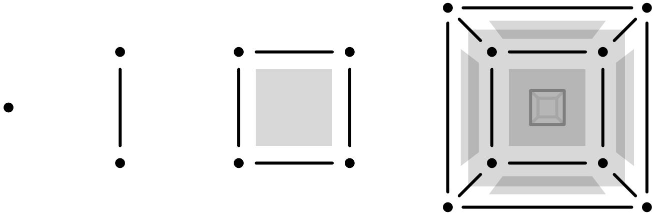
Kalai's 3^d conjecture

$$s(P) \;:=\; \cancel{f_{-1}} + f_0 + f_1 + \cdots + f_{d-1} + f_d \;=\; \#\underline{\text{non}}\text{-empty faces}$$

Conjecture. (3^d conjecture, Kalai, 1989)

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

$$s(P) \;\geq\; s(d\text{-cube}) \;=\; 3^d.$$



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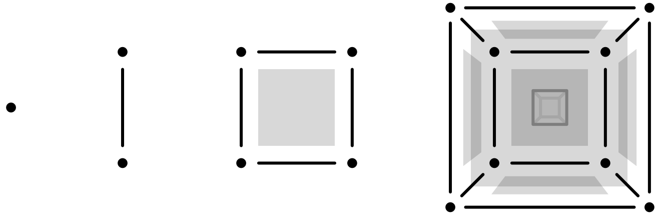


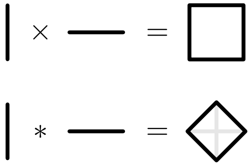
But: cube is not the only minimizer! $\rightarrow$ **Hanner polytopes**

Hanner polytopes

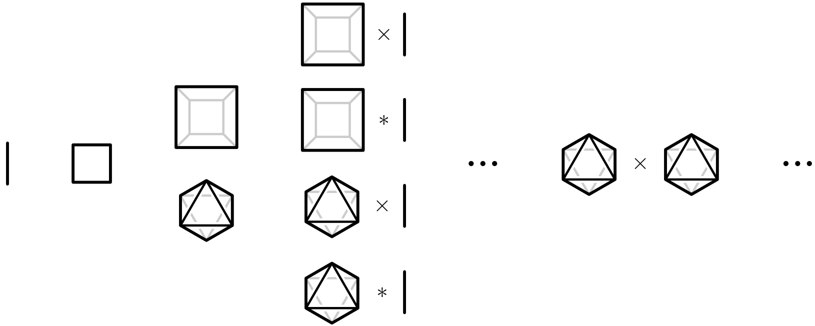
Hanner polytopes are defined recursively:

(i) start from a line segment.

(ii) recursively apply Cartesian products ($\times$) and sums ($*$)



Hanner polytopes



$\#$Hanner polytopes for $d \geq 1 \; = \; 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, ...$

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But: cube is not the only minimizer! $\to$ **Hanner polytopes**

What is known ... ?

- dimension $d \leq 3$ ✓ easy
- dimension $d = 4$ ✓ not so easy (Sanyal, Werner, Ziegler; 2007)
- simple/simplicial polytopes ✓ needs a lot of algebra
- coordinate-symmetric polytopes ✓ (Sanyal, W.; 2024)
- without requiring central symmetry ✓ easy $\to$ $s(d\text{-simplex}) = 2^d - 1$

Mahler's conjecture

Mahler volume ... $M(P) := \mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ)$

Conjecture. (3^d conjecture, Mahler, 1939)

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

measures "roundness" $\longrightarrow$ $M(P) \geq M(d\text{-cube}) = \dfrac{4^d}{d!}.$

But: cube is not the only minimizer! $\to$ **Hanner polytopes**

What is known ... ?

- dimension $d \leq 3$ ✓ not so easy ($d = 2$: 1939, $d = 3$: 2020)
- dimension $d = 4$? out of reach
- Hanner polytopes are local minimizers ✓ (Kim; 2014)
- coordinate-symmetric bodies ✓ (Meyer; 1986)
- without requiring central symmetry ? open $\to$ $M(d\text{-simplex}) = \frac{(d+1)^{d+1}}{(d!)^2}$

Kalai's flag conjecture

$$S(P) := \#\text{flags of } P$$

Conjecture. (flag conjecture, Kalai, 1989)

For every centrally symmetric d-polytope $P \subset \mathbb{R}^d$ holds

$$S(P) \geq S(d\text{-cube}) = d!\,2^d.$$

But: cube is not the only minimizer! $\to$ **Hanner polytopes**

What is known ... ?

- dimension $d \leq 3$? probably easy
- dimension $d = 4$? open
- coordinate-symmetric polytopes ✓ (Chor; 2025)

The case for 2-level polytopes

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A good class of polytopes for studying these conjectures satisfies

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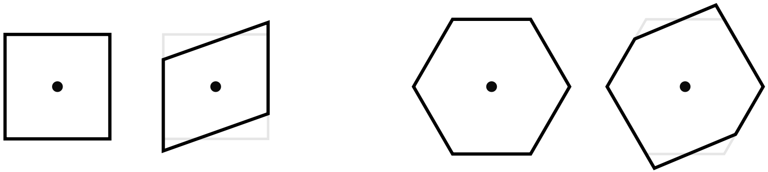
Linearly unique polytopes

Let $\mathrm{Real}(P)$ be the space of cs realization of P module linear transformations.

Definition.

A centrally symmetric polytope is

- **linearly unique** if $\mathrm{Real}(P)$ consists of a single point.
- **linearly discrete** if $\mathrm{Real}(P)$ consists of finitely many points.
- **linearly compact** if $\mathrm{Real}(P)$ is compact.



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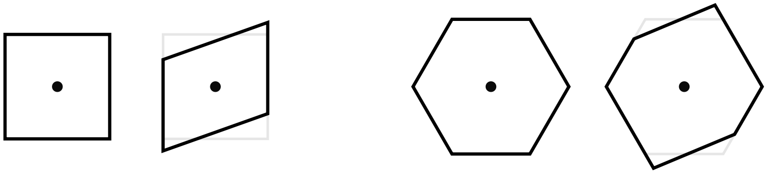

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Linearly compact polytopes

Lemma.

If P is a cs minimizer of face number, then P is linearly compact.

Proof sketch.

- if P is not linearly compact, then there is a convergent sequence $P_1, P_2, P_3, \ldots$ of realizations of P with $\lim P_n$ not being a realization of P.
- observe that in the limit, there cannot be new faces, but faces must have vanished.

$\implies$ P cannot have been a minimizer. $\square$

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Conjecture.

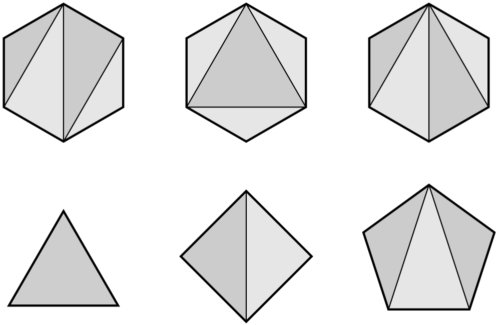
The only polytopes with compact realization spaces are linearly discrete.

- true for $d \leq 3$.
- true for matroids and oriented matroids.
- polytope realization spaces are unions of oriented matroid realization spaces.

Mahler's conjecture

$$M(P) = \mathrm{vol}(P) \cdot \mathrm{vol}(P^\circ) \geq \frac{4^d}{d!}$$

Volume and triangulation



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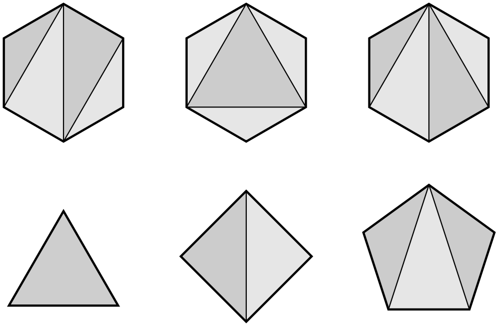

Theorem.

In a 2-level polytopes ...

- *each simplex in a <u>pulling triangulation</u> has the same volume.*
  (lattice volume 1)

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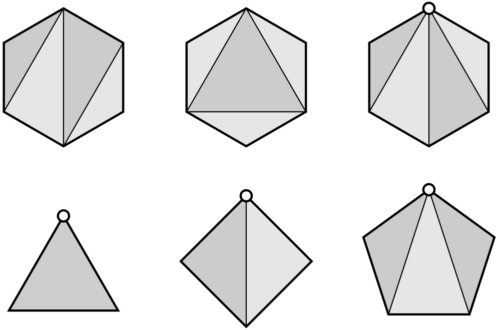

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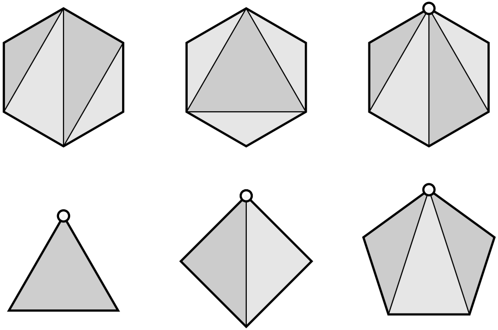

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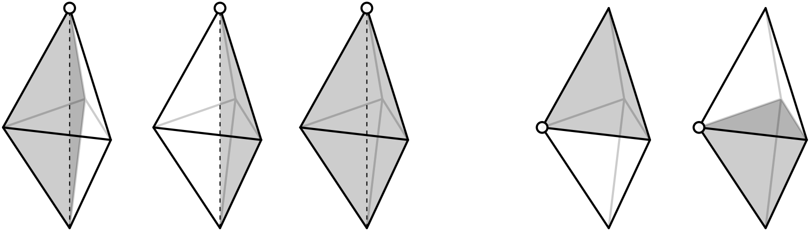

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Mahler volume and pulling triangulations

$$f_d^*(P) \text{ ... } \# \text{ simplices in pulling triangulation of } P$$

The Mahler conjecture is equivalent to the following:

Conjecture. (Sanyal, Stricker, W.)

For a centrally-symmetric 2-level polytope $P \subset \mathbb{R}^d$ holds

$$f_d^*(P) \cdot f_d^*(P^\circ) \;\geq\; d!2^{d-1}$$

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Conjecture. (Freij, Schymura, Schmitt, Ziegler; ca. 2011)

For a centrally-symmetric polytope $P \subset \mathbb{R}^d$ holds

$$M(P) \leq \frac{2^d}{(d!)^2} S(P).$$

Pulling triangulations

Theorem.

If P is a 2-level polytope, then

$$f_d^*(P) = \sum_{F_0 \preceq \cdots \preceq F_d} \left(1 - \frac{f_0(F_0)}{f_0(F_1)}\right) \cdots \left(1 - \frac{f_0(F_{d-1})}{f_0(F_d)}\right).$$

Proof.

- For a vertex v and facet F_{d-1}, let $[v \notin F_{d-1}]$ denote the indicator function.
- We have
$$f_d^*(P) = \sum_{F_{d-1}} [v \notin F_{d-1}] f_{d-1}^*(F_{d-1}).$$
- Take expectation value *w.r.t.* a uniform random choice of v:

$$f_d^*(P) = \sum_{F_{d-1}} \left(1 - \frac{f_0(F_{d-1})}{f_0(P)}\right) f_{d-1}^*(F_{d-1}) = \sum_{F_0 \preceq \cdots \preceq F_d} \left(1 - \frac{f_0(F_0)}{f_0(F_1)}\right) \cdots \left(1 - \frac{f_0(F_{d-1})}{f_0(F_d)}\right)$$

$\square$

Computational results

dim	0	1	2	3	4	5	6	7	8
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For $d \leq 8$ we found

- *no* counterexample to Kalai's 3^d conjecture
- *no* counterexample to Kalai's flag conjecture
- *no* counterexample to Mahler's conjecture
- *no* counterexample to the flag-volume bound
- in all cases, equality is attained only by Hanner polytopes.

Thank you.