Rigidity and Reconstruction of Convex Polytopes

– an application of Wachspress Geometry –

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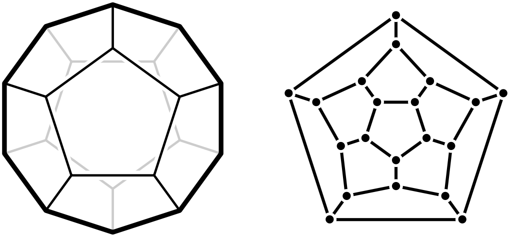
08. January, 2025

The setting: convex polytopes

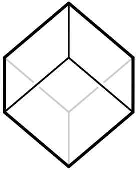
$$P = \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$$

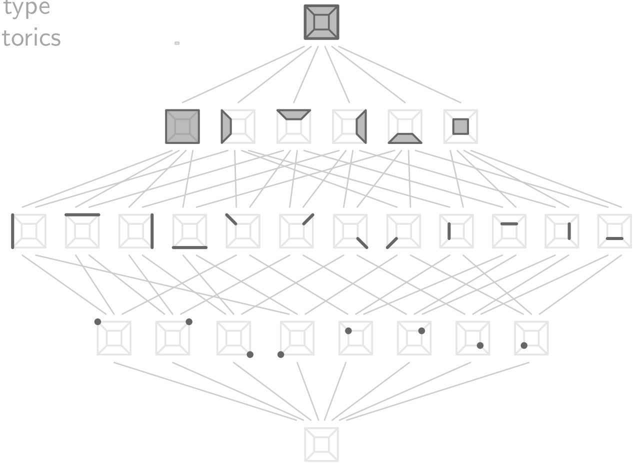


- always convex
- general dimension $d \geq 2$
- general geometry & combinatorics (not only simple/simplicial/lattice/...)
- always of full dimension

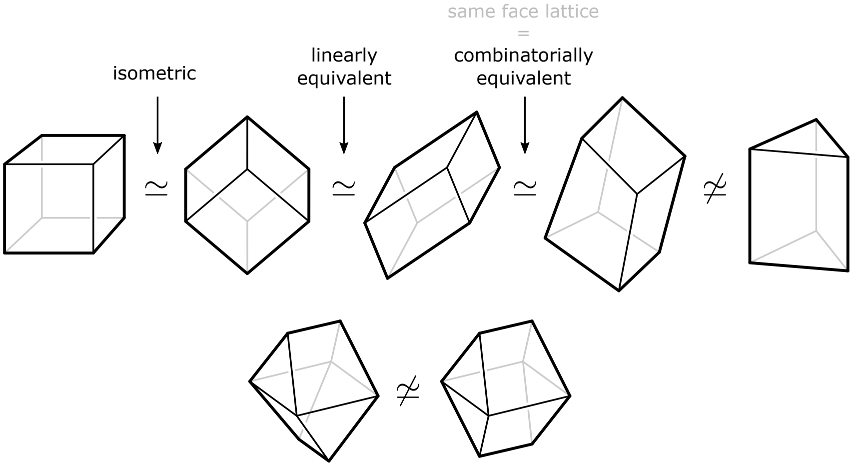
The combinatorics of a polytope

face lattice
$\cong$ combinatorial type
$\cong$ (full) combinatorics

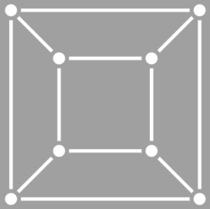


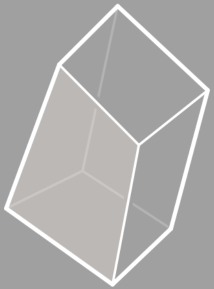


Geometric realizations



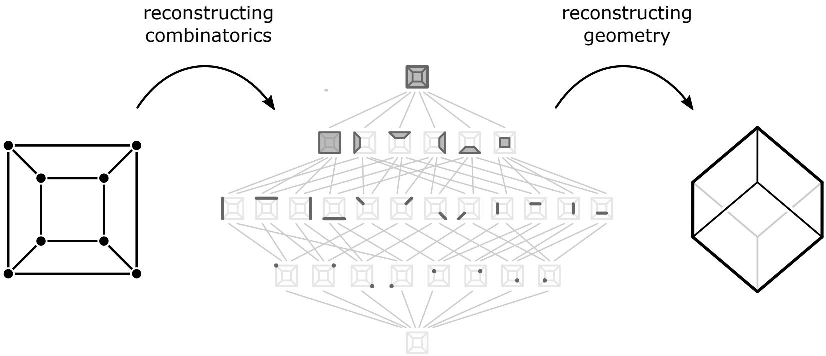
Reconstruction of polytopes



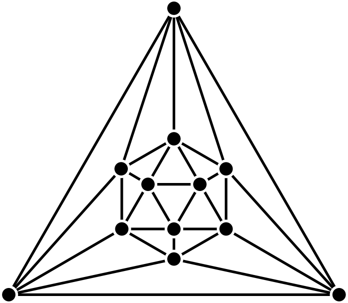


Reconstruction of polytopes

"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"



Reconstructing combinatorics ($d = 3$)



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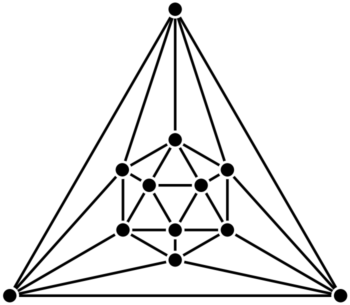

Question I: Is this the edge graph of a polyhedron? (Steinitz problem)

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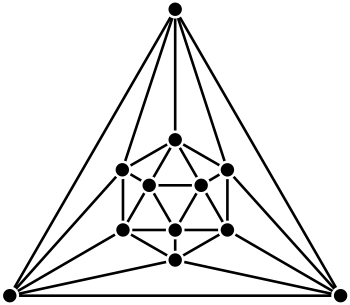

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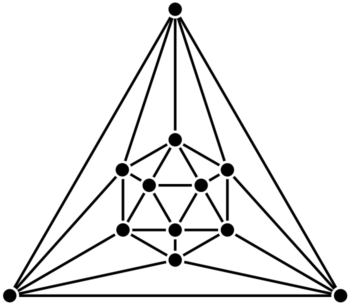

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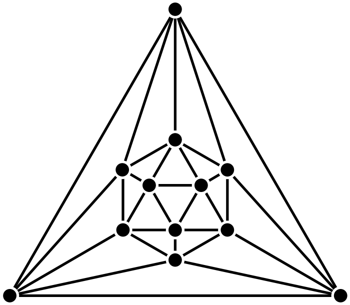

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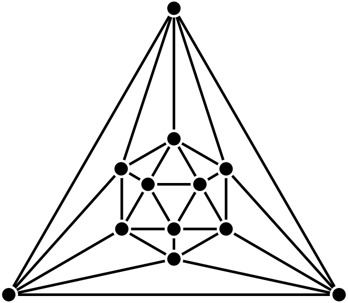

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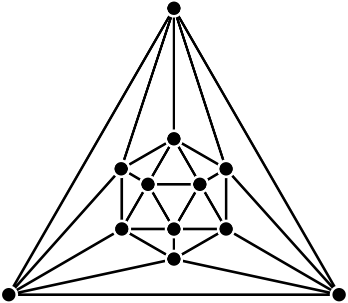

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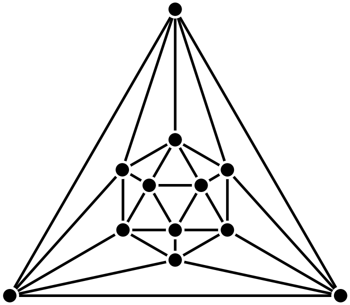

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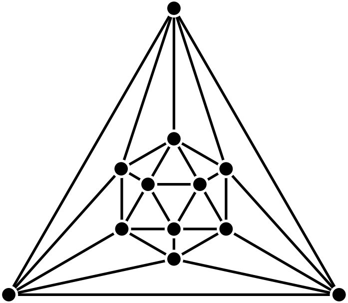

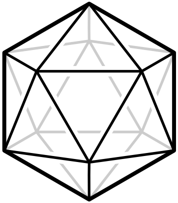

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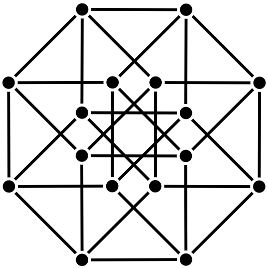

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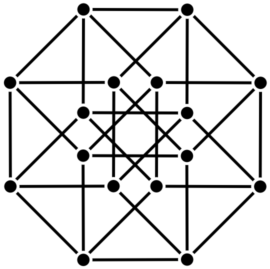

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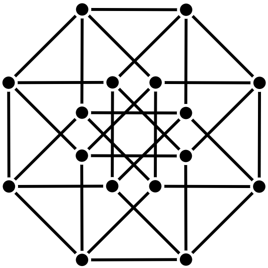

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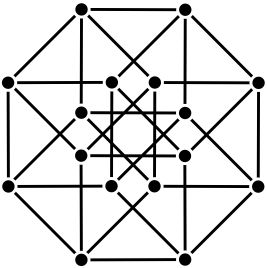

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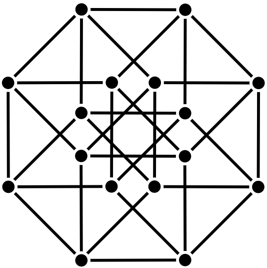

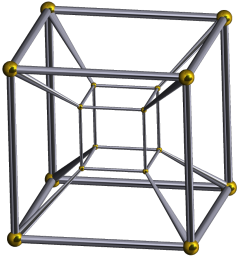

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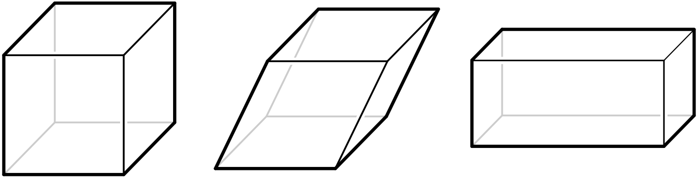
Reconstructing geometry

Given the full combinatorics, can we reconstruct from ...

- edge lengths ✗
- dihedral angles ✗



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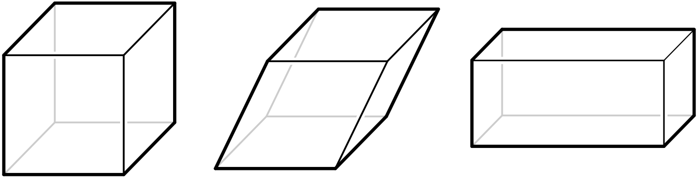

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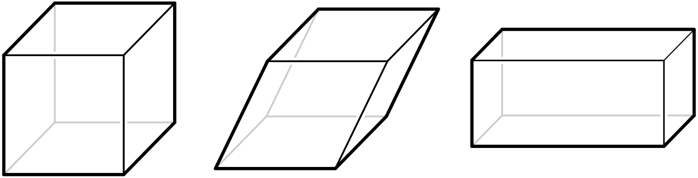
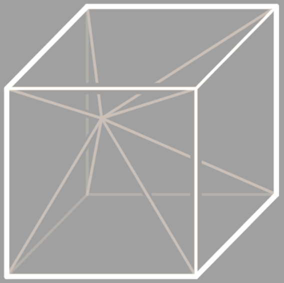


Cauchy's rigidity theorem (Cauchy, 1813)

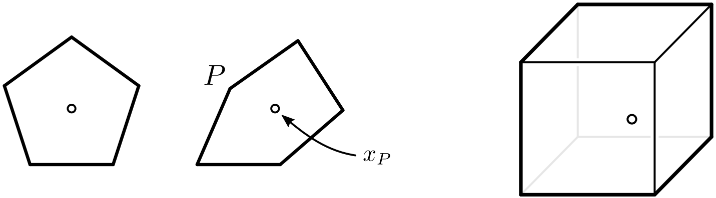
A polytope is uniquely determined (up to isometry) by its combinatorics and the shapes of its 2-faces.

Reconstruction of pointed polytopes



Pointed polytopes

:= polytope $P \subset \mathbb{R}^d$ + point $x_P \in \mathbb{R}^d$



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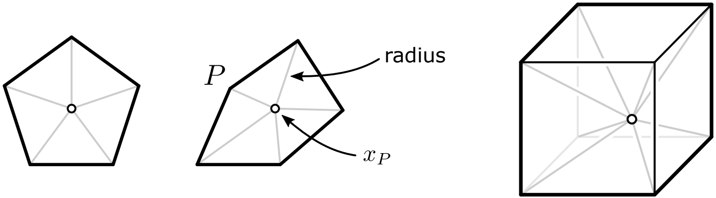

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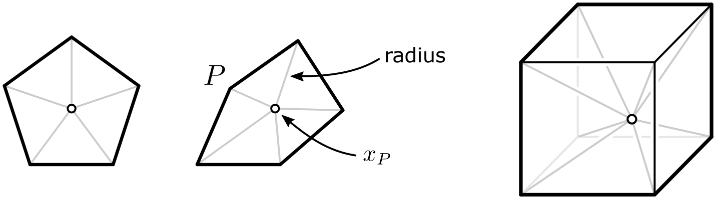

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- Is a pointed polytope determined by the graph, edge lengths and radii?

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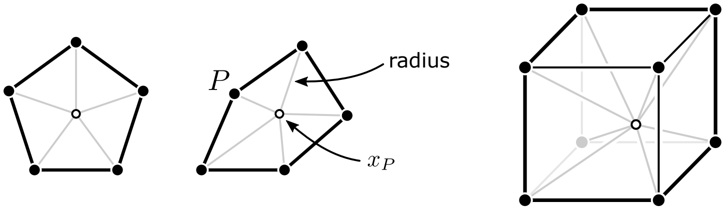

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- ... also as a framework? (coned polytope frameworks)

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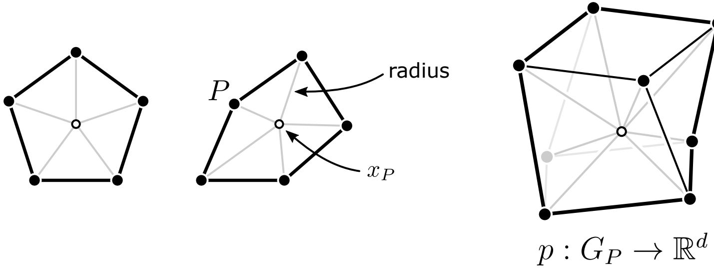

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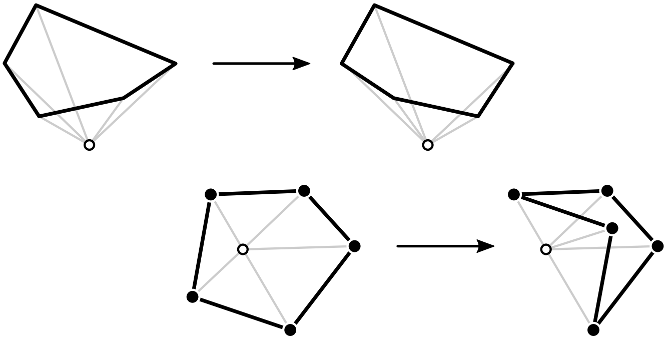
Pointed polytopes and frameworks



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Conjecture. (tensegrity version)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge graph and

(i) $x_Q \in \operatorname{int}(Q)$

(ii) *edges in Q are at most as long as in P,*

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then P and Q are isometric.

"A polytope cannot become larger if all its edges become shorter."

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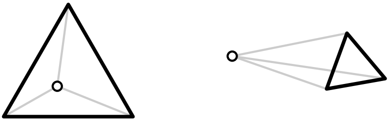

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Conjecture holds in special cases (W., 2023)

I. Q is a small perturbation of P

- one can replace Q by a graph embedding $q\colon G_P \to \mathbb{R}^d$
- $\cong$ locally rigid as a framework

II. P and Q are centrally symmetric

- one can replace Q by a centrally symmetric graph embedding $q\colon G_P \to \mathbb{R}^e$
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III. P and Q are combinatorially equivalent

- in particular true for polytope of dimension $d \leq 3$

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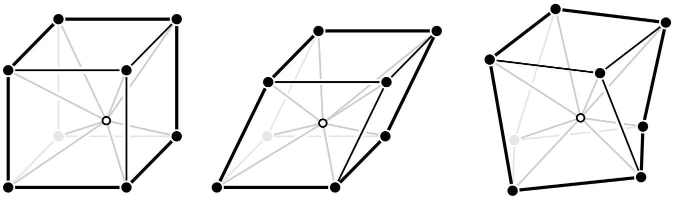

$P \subset \mathbb{R}^d$ $\quad$ $Q \subset \mathbb{R}^e$ $\quad$ $q : G_P \to \mathbb{R}^e$

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Is this surprising?

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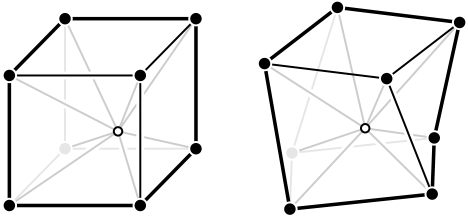
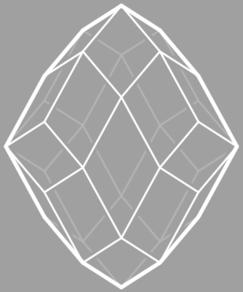
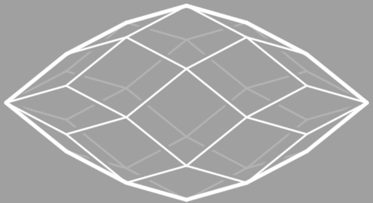


$$\#\text{DOFs} - \#\text{constraints} = (\overset{V}{8} + 1) \times \overset{d}{3} - (\overset{E}{12} + \overset{V}{8}) = 7 = 6 + 1.$$

Ingredients to the proof





WARMUP: SIMPLICES

$P, Q \subset \mathbb{R}^d$ pointed simplices with $x_P = x_Q = 0$,

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$$\rotatebox{90}{$\leq$}\ \text{(ii)}$$

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Warmup: simplices

$P, Q \subset \mathbb{R}^d$ pointed simplices with $x_P = x_Q = 0$,

(i) $0 \in \mathrm{int}(Q)$, $\implies 0 = \sum_i \alpha_i q_i$ ... convex combination with $\alpha_i > 0$

(ii) edges in Q are at most as long as in P.

(iii) radii in Q are at least as large as in P.

Proof.

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Polytopes energy

Fix $\alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$

$$\alpha\text{-}\textbf{energy:} \qquad E_\alpha(P) := \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

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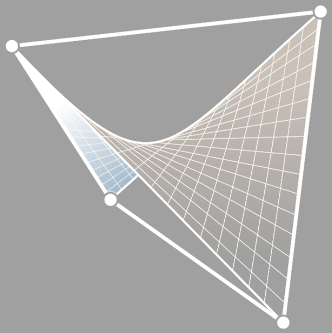
Key theorem

Let α be the Wachspress coordinates of some interior point of P. If edges in $q : G_P \to \mathbb{R}^e$ are not longer than in P, then

$$E_\alpha(q) \leq E_\alpha(P),$$

with equality if and only if $q \simeq_{\text{affine}} P$.

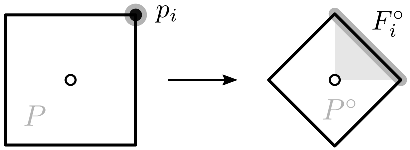
A glimpse of Wachspress geometry



Wachspress coordinates are ...

I. ... relative cone volumes (Ju et al., 2005)

polar dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, p_i \rangle \leq 1 \text{ for all } i \in V(G_P)\}$.



$$\alpha_i = \frac{\mathrm{vol}(F_i^\circ)}{\|p_i\| \,\mathrm{vol}(P^\circ)}$$

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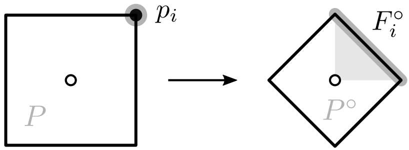

$$\alpha_i = \frac{\mathrm{vol}(F_i^\circ)}{\|p_i\| \,\mathrm{vol}(P^\circ)}$$

II. ... the unique rational GBCs of lowest possible degree (Warren, 2003)

$$\alpha_i(x) = \frac{\mathrm{p}_i(x)}{\mathrm{q}(x)}$$ where $\mathrm{q}(x) = \sum_i \mathrm{p}_i(x)$ ... adjoint polynomial

Theorem. (Warren)

Wachspress coordinates are the unique rational GBCs of lowest possible degree.

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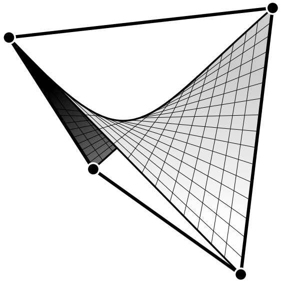

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Wachspress coordinates are the unique rational GBCs of lowest possible degree.

Wachspress coordinates are …

III. … a "shadow" of a higher rank objects

Theorem. (Izmestiev, 2007)

For a pointed polytope $P \subset \mathbb{R}^d$ there is a matrix $M \in \mathbb{R}^{n\times n}$ with

(i) $M_{ij} > 0$ *whenever* $ij \in E(G_P)$,

(ii) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E(G_P)$,

(iii) $\dim\ker(M) = d$,

(iv) $MX_P = 0$, *where* $X_P^\top = (p_1, ..., p_n) \in \mathbb{R}^{d\times n}$,

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$$\alpha_i(x) = \sum_j M_{ij}(x)$$ (W., 2023)

Proving the key theorem ...

Key theorem

Let α be the Wachspress coordinates of some interior point of P. If edges in $q : G_p \to \mathbb{R}^e$ are not longer than in P, then

$$E_\alpha(q) \le E_\alpha(P).$$

"The skeleton of P has the maximal α-energy among all embeddings of G_P whose edges are not longer than in P."

$$\begin{aligned} \max \quad & E_\alpha(q) \\ \text{s.t.} \quad & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{aligned}$$

Proof via semidefinite programming

$$\begin{array}{rl} \max & E_\alpha(q) \\ \text{s.t.} & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{array}$$

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$$\Downarrow \quad \text{by translation invariance}$$

$$\begin{array}{rl} \max & \sum_i \alpha_i \|q_i\|^2 \\ \text{s.t.} & \sum_i \alpha_i q_i = 0 \\ & \|q_i - q_j\| \le \|p_i - p_j\|, \quad \text{for all } ij \in E \\ & q_1, ..., q_n \in \mathbb{R}^n \end{array}$$

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$\Downarrow$ dual program

$$\begin{aligned} \min \quad & \textstyle\sum_{ij \in E} w_{ij} \|p_i - p_j\|^2 \\ \text{s.t.} \quad & L_w - \operatorname{diag}(\alpha) + \mu \alpha \alpha^\top \succeq 0 \\ & w \ge 0, \mu \text{ free} \end{aligned}$$

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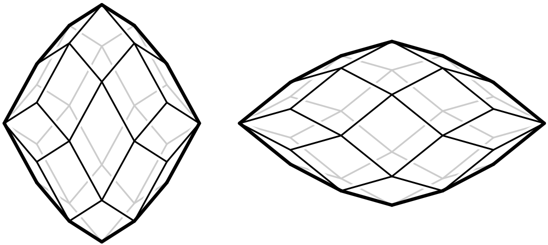
CONSEQUENCES

Corollary.

A pointed polytope is uniquely determined (up to affine transformations) by its edge graph, edge lengths and Wachspress coordinates.



A polytope can be reconstructed in polynomial time (via a semidefinite program).

Are we done ... ?

$P, Q \subset \mathbb{R}^d$ pointed polytopes with $x_P = x_Q = 0$,

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Proof.

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$\wedge\!|$ (iii) $\qquad \vee\!|$ (i) $\qquad \vee\!|$??

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Therefore $P \simeq Q$. □

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+ Wachspress coordinates in P

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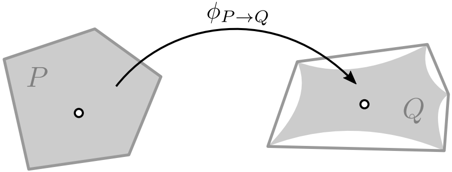
The Wachspress map $\phi\colon P \to Q$

$$x \in P \;\longmapsto\; \alpha(x) \in \Delta_n \;\longmapsto\; \phi(x) := \sum_i \alpha_i(x) q_i \in Q$$



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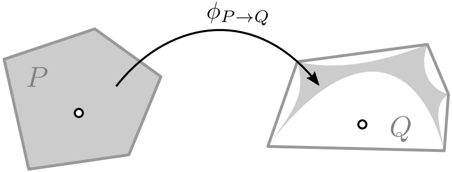

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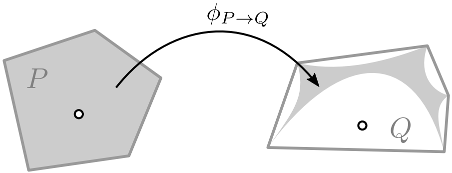


Key lemma.

If $P \subset \mathbb{R}^d$ and $q : G_P \to \mathbb{R}^e$ satisfy

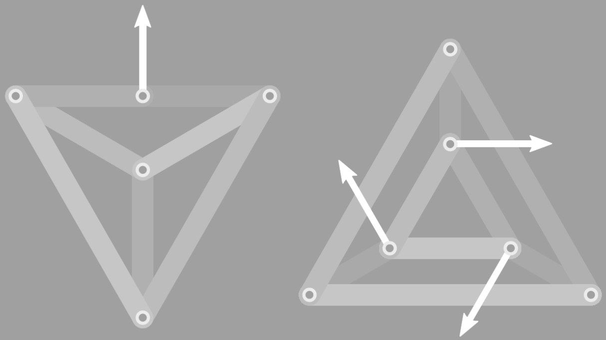
(i) *there is $x \in \mathrm{int}(P)$ with $\|\phi(x)\| \le \|x\|$,* (*e.g. if $\phi(x) = 0$*)

(ii) *edges in q are at most as long as in P,*

(iii) *radii in q are at least as large as in P,*

then $q \simeq_{\text{iso}} P$.

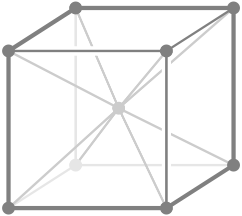
SECOND-ORDER RIGIDITY & OTHER CONJECTURES



First- and second order rigidity

Coned polytope frameworks are ...

✓ rigid



First- and second order rigidity

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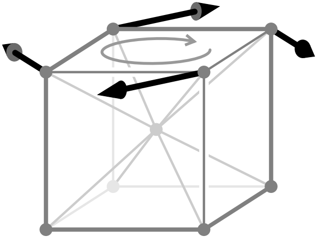
- ✗ not first-order rigid



First- and second order rigidity

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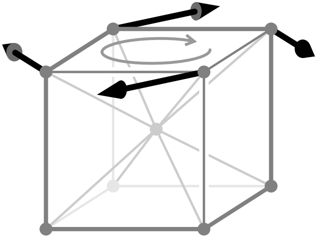
- ✗ not first-order rigid
- ? *probably* second-order rigid



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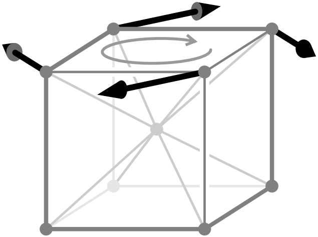

Conjecture. (Connelly, Gortler, Theran, W.; 2024)

Coned polytope frameworks are second-order rigid. (actually prestress stable)

This is implied by the following conjecture of independent interest ...

A stronger conjecture

Minkowski's
balancing condition

$$0 = \sum_i V_i n_i \qquad .$$

A stronger conjecture

Minkowski's
balancing condition

$$0 = \sum_i V_i n_i \quad \Longrightarrow \quad 0 = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i V_i n_i = \sum_i \dot{V}_i n_i + \sum_i V_i \dot{n}_i.$$

A stronger conjecture

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Conjecture.

If there is no first-order change in the angles between adjacent facets, then

$$\sum_i \dot{V}_i n_i = \sum_i V_i \dot{n}_i = 0.$$

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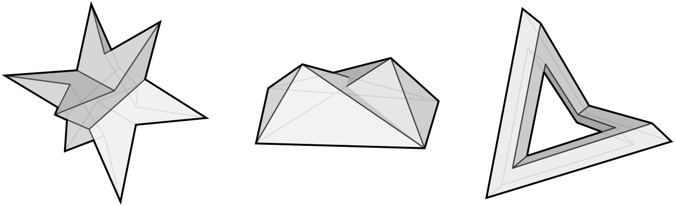
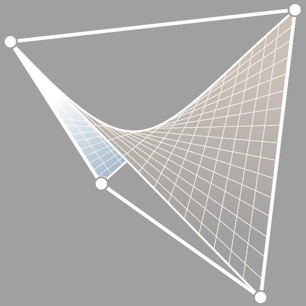


Thank you.



M. Winter, *"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"* (2023)

R. Connelly, S.J. Gortler, L. Theran, M. Winter
"Energies on coned convex polytopes" (2024)
"The stress-flex conjecture" (2024)