Adjoint degrees and scissors congruence for polytopes

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(joint work with Tom Baumbach, Ansgar Freyer and Julian Weigert)

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES

combinatorial synergies

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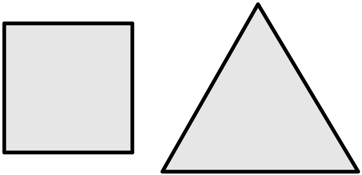
Scissors congruence

Two polytopes P and Q are **scissors congruent** if

$$P = P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n \qquad Q = Q_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} Q_n.$$

with $Q_i = S_i(P_i)$, where $S_i \in \mathrm{Iso}(\mathbb{R}^d)$ are isometries.



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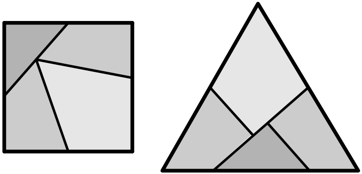

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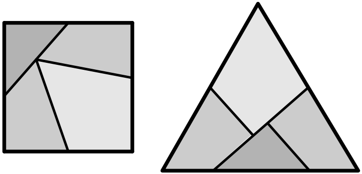

Theorem (Wallace, Bolyai, Gerwien; 1807/33/35)

Two polygons P, Q are scissors congruent if and only if $\mathrm{vol}(P) = \mathrm{vol}(Q)$.

Hilbert's Third Problem

Given any two polyhedra P and Q of equal volume, is it always possible to dissect P into finitely many polyhedral pieces $P_1, ..., P_n$, which can then be reassembled to yield Q?

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Theorem. (Dehn; 1901)

If $P, Q \subset \mathbb{R}^3$ are scissors congruent, then they have the same Dehn invariant.

$$D(P) := \sum_{e \subset P} \ell_e \otimes_{\mathbb{Z}} \theta(e)/2\pi \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}.$$

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Theorem. (Sydler; 1965)

$P, Q \subset \mathbb{R}^3$ are scissors congruent if and only if they have the same volume and the same Dehn invariant.

Valuations

Whenever P, Q, $P \cap Q$ and $P \cup Q$ are polytopes, a **valuation** satisfies

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$$

Examples:

- volume
- Dehn invariant
- surface area measure
- Euler characteristic
- mixed volumes
- number of contained lattice points

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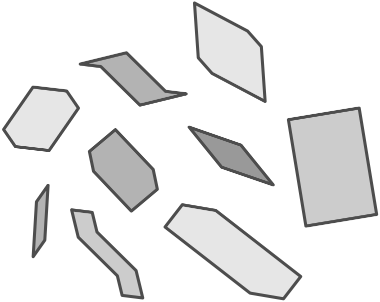
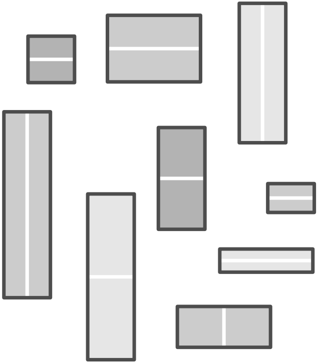
What we mainly care about (true for *simple valuations*):

$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

Two composition puzzles





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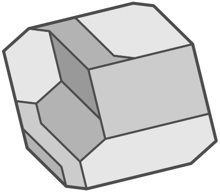

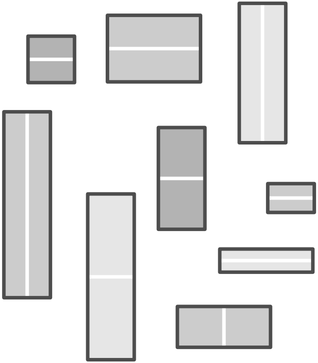

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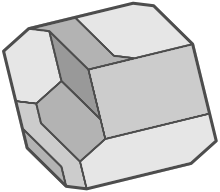
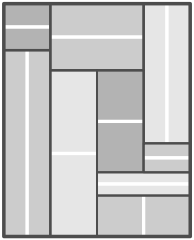
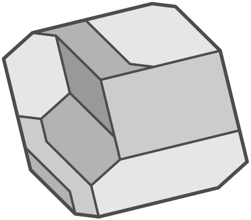




Puzzle I

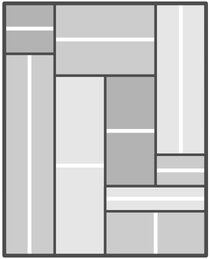


Let $\nu(P)$ be the *surface area measure* of $P \subset \mathbb{R}^d$ on $\mathbb{S}^{d-1}$.

$$\phi(P) := \nu(P) - \nu(-P)$$

Fact: a convex polygon P is centrally symmetric if and only if $\phi(P) = 0$.

Puzzle II



$$\phi(P) := \int_{I_1 \times I_2} e^{2\pi i(x_1+x_2)}\,\mathrm{d}x = \int_{I_1} e^{2\pi i x_1}\,\mathrm{d}x_1 \cdot \int_{I_2} e^{2\pi i x_2}\,\mathrm{d}x_2$$

Fact: a rectangle P has an integer side length if and only if $\phi(P) = 0$.

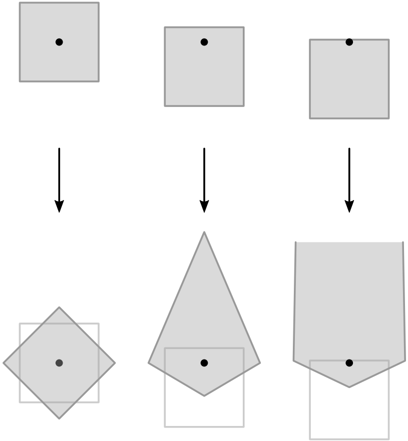
$\longrightarrow$ Stan Wagon, *"Fourteen Proofs of a Result About Tiling a Rectangle"*

Dual volumes and the canonical form

Ω

Polar duality

(polar) dual ... $P^\circ := \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq 1 \text{ for all } y \in P\}.$



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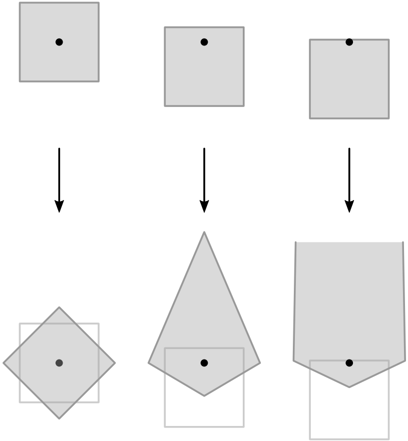

Central new idea: the volume of the dual behaves valuative!

Dual volumes

canonical form...

$$\Omega(P;x) := d!\,\mathrm{vol}(P-x)^\circ = \frac{p(x)}{q(x)}$$

Observe: this is a rational function in x.

$\implies$ Ω can be extended to points x outside of P.

Theorem. (Arkani-Hamed, Bai, Lam; 2017)

$$\Omega(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n; x) = \Omega(P_1;x) + \cdots + \Omega(P_n;x).$$

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$$\Omega(P;x) \cdot \prod_F L_F(x) = r(x)$$

- $L_F(x) := h_F - \langle u_F, x\rangle$ ... facet-defining linear form
- u_F ... unit normal vector
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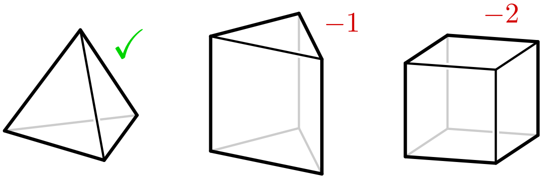

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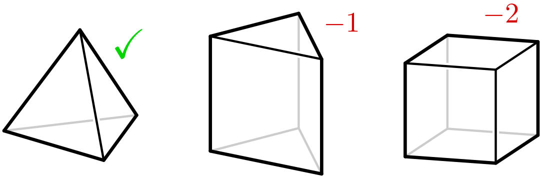

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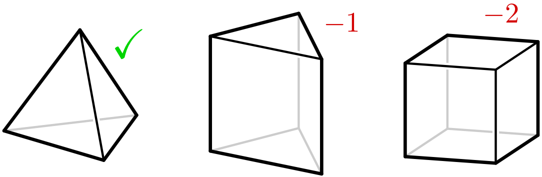

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Example: for the d-cube $\square_d := [-1,1]^d$ we have

$$\Omega(\square_d; x) = \frac{\text{some constant}}{\prod_i (1 - x_i^2)} \quad\Longrightarrow\quad \mathrm{drop}(\square_d) = d - 1.$$

The drop under composition

Lemma.

$$\mathrm{drop}(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) \geq \min_i \mathrm{drop}(P_i).$$

Proof. Observe

$$\deg \Omega(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \deg \Big(\sum_i \Omega(P_i) \Big) \leq \max_i \deg \Omega(P_i).$$

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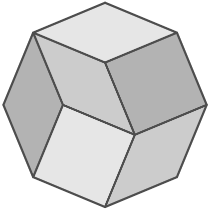

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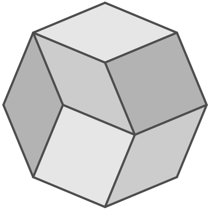

Questions:

- What other polytopes have a drop?
- What characterizes polytopes with a particular drop s?

Properties of the drop

(i) $\mathrm{drop}(P_1 \times \cdots \times P_n) = n - 1 + \sum_i \mathrm{drop}(P_i)$.

(ii) if F is a facet of P, then

$$\mathrm{drop}(F) \geq \mathrm{drop}(P) - 1,$$

with equality if and only if P has a facet F' parallel to F.

(iii) $\mathrm{drop}(P) \leq d - 1$.

(iv) $\mathrm{drop}(SP + t) = \mathrm{drop}(P)$.

(v) if π is a projection onto a hyperplane, then

$$\mathrm{drop}(\pi P) \geq \mathrm{drop}(P) - 1.$$

(vi) $\mathrm{drop}(P_1 + \cdots + P_n) \geq (d - 1) - \sum_i (d_i - 1) + \sum_i \mathrm{drop}(P_i)$.

(vii) if P is centrally symmetric

$$\mathrm{drop}(P) \text{ is } \begin{cases} \text{even} & \text{if } d \text{ is odd} \\ \text{odd} & \text{if } d \text{ is even} \end{cases}.$$

Maximal drop

Lemma.

A zonotope $P \subset \mathbb{R}^d$ *attains the maximal possible* $\operatorname{drop}(P) = d - 1$.

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A zonotope $P \subset \mathbb{R}^d$ attains the maximal possible $\operatorname{drop}(P) = d - 1$.

Proof. (actually, three proofs) We have $\operatorname{drop}(P) \le d - 1$, but also a zonotope ...

1. ... is a projection of an n-cube $\square_n$:

$$\operatorname{drop}(\pi_d \square_n) \ge \underbrace{\operatorname{drop}(\square_n)}_{=n-1} - (n - d) = d - 1.$$

2. ... is a Minkowski sum of line segments $S_1, ..., S_n$:

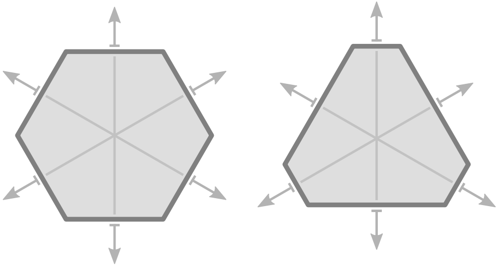
$$\operatorname{drop}(S_1 + \cdots + S_n) \ge (d-1) - \sum_i \underbrace{(\dim(S_i) - 1)}_{=0} + \sum_i \underbrace{\operatorname{drop}(S_i)}_{=0} = d - 1.$$

3. ... can be tiled by parallelepipeds $P_1, ..., P_n$:

$$\operatorname{drop}(P_1 \cup \cdots \cup P_n) \ge \min_i \underbrace{\operatorname{drop}(P_i)}_{=d-1} = d - 1.$$

What else has a drop?

Observation: for maximal drop facets must come in parallel pairs.



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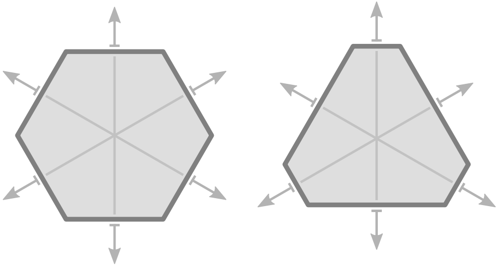

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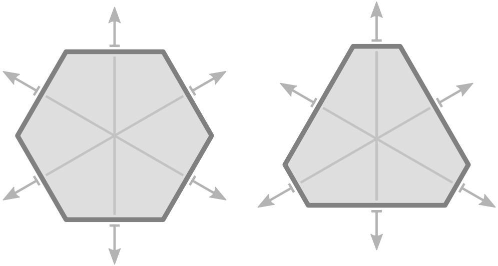

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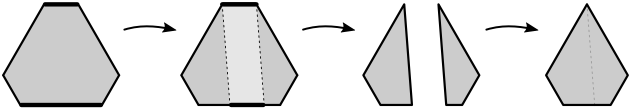
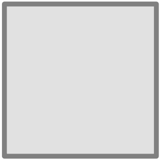
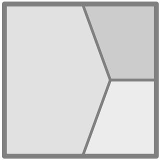
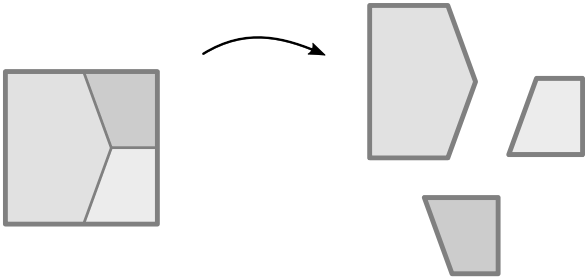
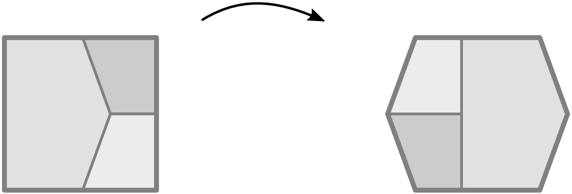
"Proof" that the answer is <u>No</u>:

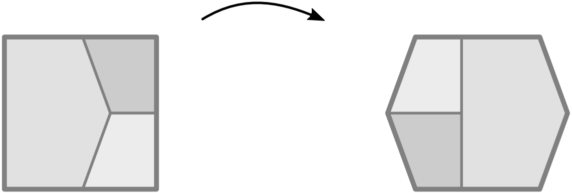




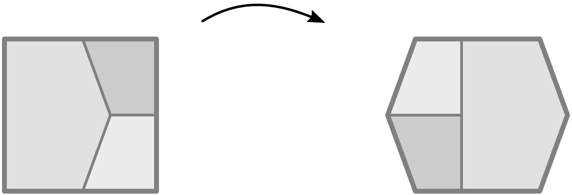








$$\begin{aligned}\phi(P) &= \phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n)\\ &= \phi(P_1) + \cdots + \phi(P_n)\\ &= \phi(P_1 + t_1) + \cdots + \phi(P_n + t_n)\\ &= \phi\big((P_1 + t_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (P_n + t_n)\big) = \phi(Q)\end{aligned}$$



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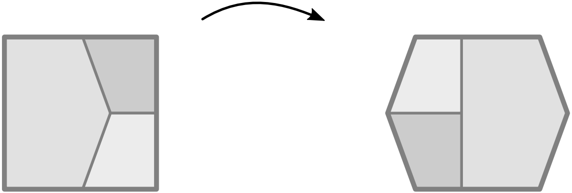
Translation scissors congruence



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$$

A new translation-invariant valuation

$$\Omega_0$$

The view from infinity

$$\Omega_0(P;x) := \Omega(P;x_0,x)|_{x_0=0} = \frac{\mathrm{adj}_P(x_0,x)|_{x_0=0}}{(-1)^m \prod_F \langle u_F, x\rangle}.$$

One can view this as

- restricting Ω to the hyperplane at infinite (given by $x_0 = 0$).
- restricting the numerator (resp. denominator) to the "expected leading monomials".

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Lemma.

Ω_0 *is a translation-invariant valuation.* *(whereas Ω is not)*

Proof idea. Translations preserve the leading coefficients of a polynomial:

$$p(x) = \sum_\alpha p_\alpha x^\alpha \quad \longrightarrow \quad p(x+t) = \sum_\alpha p_\alpha (x+t)^\alpha. \qquad \square$$

How to use Ω_0

Observation: $\Omega_0(P) = 0$ if and only if $\mathrm{drop}(P) > 0$.

Theorem.

If P and Q are translation scissors congruent, then

$$\mathrm{drop}(P) > 0 \quad \Longleftrightarrow \quad \mathrm{drop}(Q) > 0.$$

But ...

- We can only distinguish drop vs. no-drop.
- We lose all information about the precise value of the degree drop.

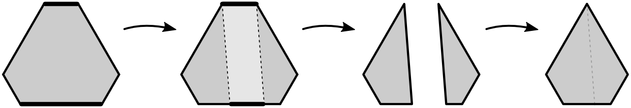
Central symmetry $\Leftrightarrow$ drop $= 1$

Theorem.

For $d = 2$ we have $\mathrm{drop}(P) > 0$ if and only if P is centrally-symmetric.

Proof.

- every edge needs a parallel edge $\implies$ must be a $2n$-gon



- $\Omega_0(P) = 0$ and this is preserved in all steps $\lightning$ $\square$

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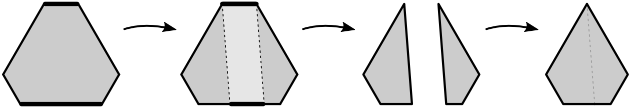

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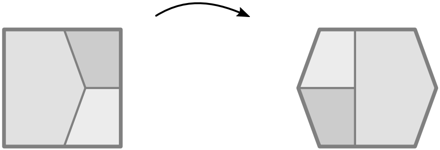

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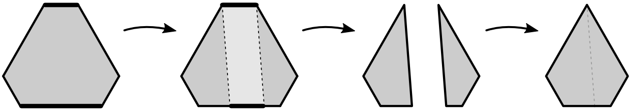


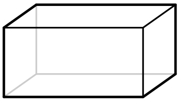
- $\Omega_0(P) = 0$ and this is preserved in all steps $\lightning$ $\square$

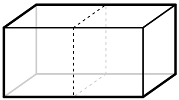
Theorem.

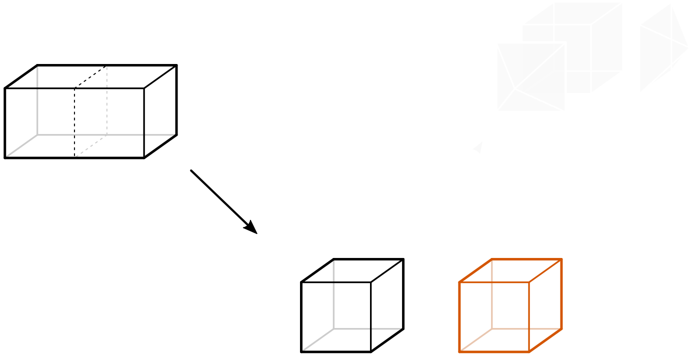
P has maximal degree drop $d-1$ if and only if P is a zonotope.

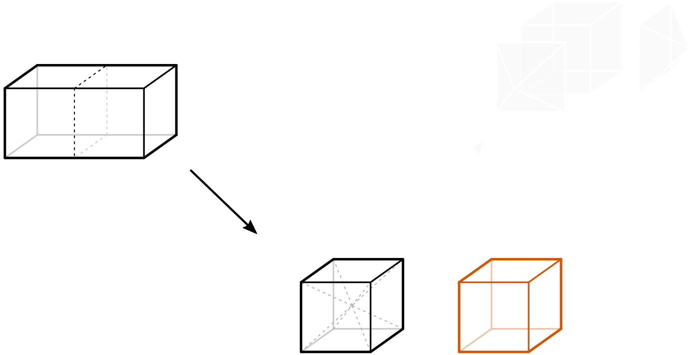
Proof.

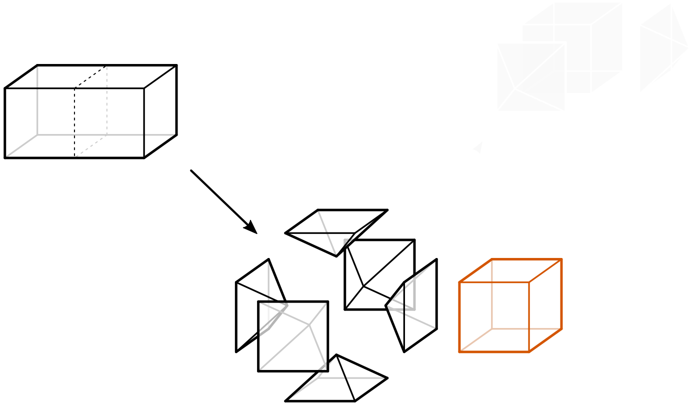
- if P has maximal drop, then so do its faces.
- all 2-faces centrally symmetric $\implies$ zonotope. $\square$

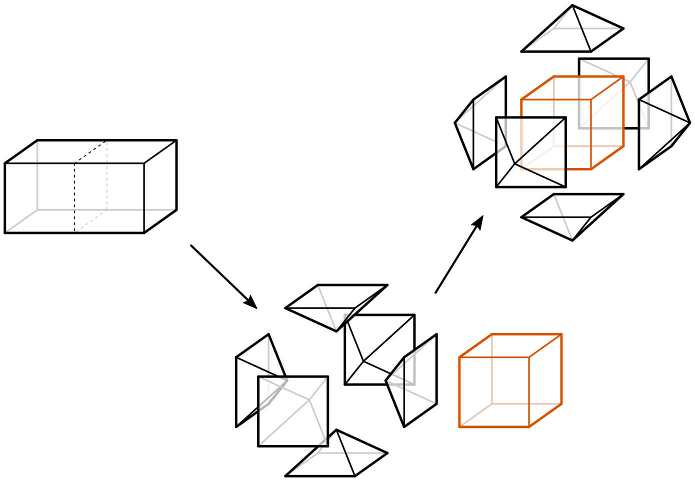


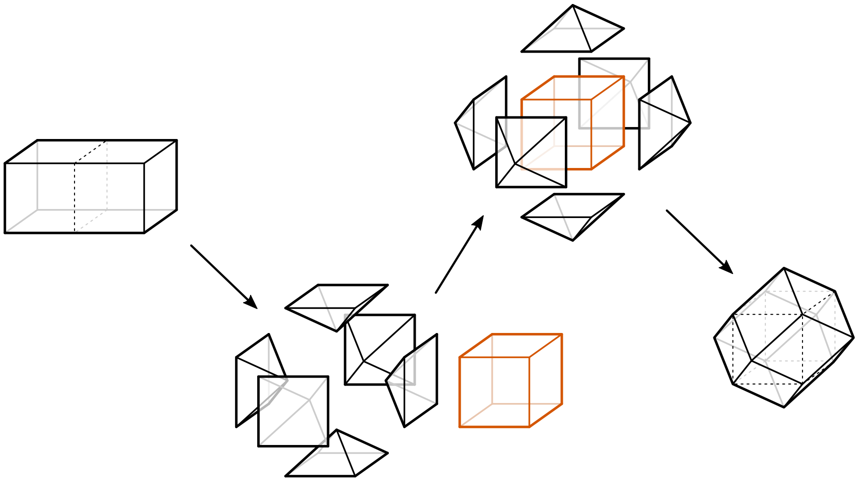


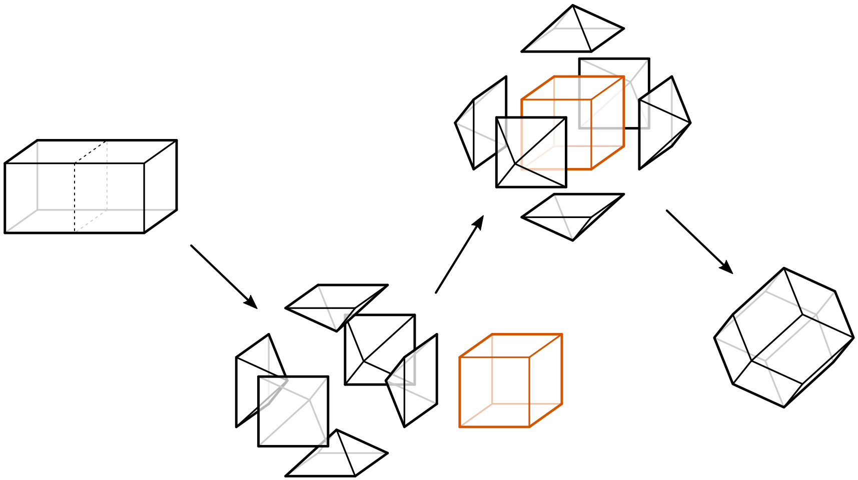


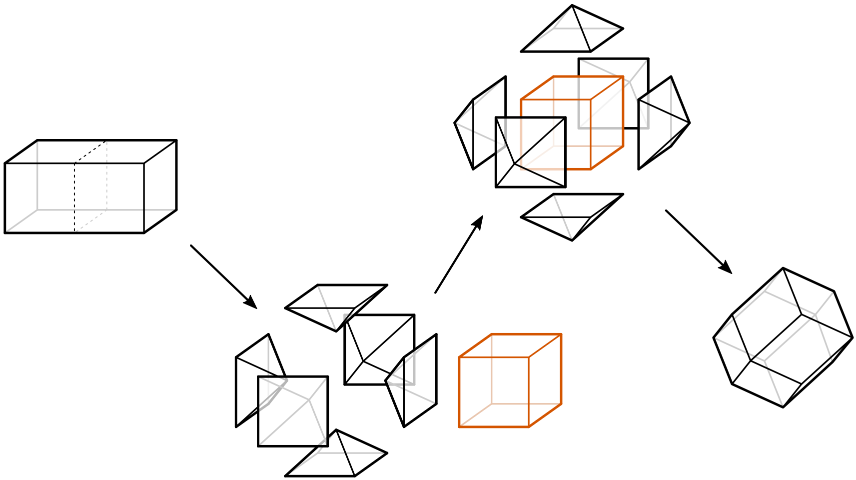












Question: Are zonotopes only translation scissors congruent to zonotopes?
or stronger, is the precise degree drop preserved under TS congruence?

Yes and no

Theorem.

In dimension $d \le 3$ the degree drop is a translation scissors invariant.

Proof. (for $d = 3$)

$$\mathrm{drop}(P) = \begin{cases} 0 & \Omega_0 \neq 0 \\ 1 & \Omega_0 = 0 \text{ and } P \text{ is not centrally symmetric} \\ 2 & \Omega_0 = 0 \text{ and } P \text{ is centrally symmetric} \end{cases}.$$

Both $\Omega_0 = 0$ and being centrally symmetric are TS invariant. □

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This is not true in dimensions $d \geq 4$.

Example: 4-cube and 24-cell.

Homogeneity

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A valuation is ***k*-homogeneous** if for all $\lambda > 0$ holds

$$\phi(\lambda P) = \lambda^k \phi(P).$$

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Lemma.

Ω_0 is 1-homogeneous. *(whereas Ω is not)*

Proof.

$$\begin{aligned}\Omega(\lambda P; x) &= \mathrm{vol}(\lambda P - x)^\circ \\ &= \mathrm{vol}(\lambda(P - x/\lambda))^\circ \\ &= \mathrm{vol}(\lambda^{-1}(P - x/\lambda)^\circ) \\ &= \lambda^{-d}\,\mathrm{vol}(P - x/\lambda)^\circ = \lambda^{-d}\,\Omega(P; x/\lambda).\end{aligned}$$

$$\begin{aligned}\Omega_0(\lambda P; x) &= \lambda^{-d}\,\Omega(P; 0, x/\lambda) = \lambda^{-d}\frac{\mathrm{adj}_P(0, x/\lambda)}{\prod_F L_F(0, x/\lambda)} \\ &= \lambda^{-d}\frac{\lambda^{-(m-d-1)}\,\mathrm{adj}_P(0,x)}{\lambda^{-m}\prod_F L_F(0,x)} = \lambda\frac{\mathrm{adj}_P(0,x)}{\prod_F L_F(0,x)} = \lambda\,\Omega_0(P; x).\end{aligned}$$

$\square$

Homogeneity is great!

Theorem. (McMullen)

If Ω_0 is 1-homogeneous, then it is **Minkowski additive***:*

$$\Omega_0(P_1 + \cdots + P_n) = \Omega_0(P_1) + \cdots + \Omega_0(P_n).$$

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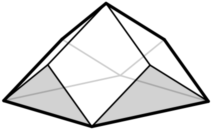
Observation: Minkowski sums of low-dimensional polytopes have a degree drop.



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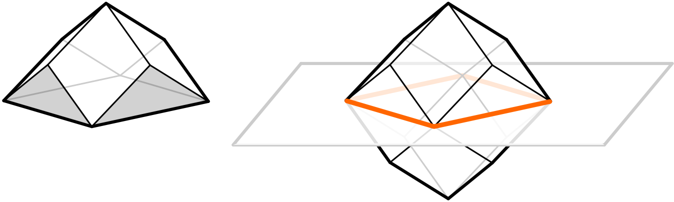

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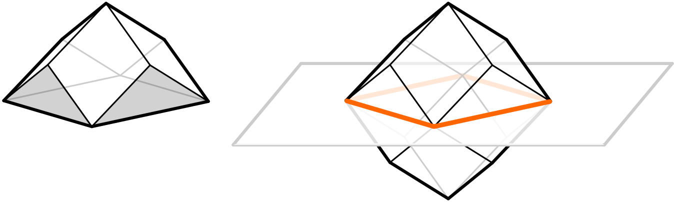

Theorem.

If P is a centrally-symmetric polytope of odd dimension with $\mathrm{drop}(P) > 0$, then each half Q of a central dissection has $\mathrm{drop}(Q) > 0$ as well.

A characterization in dimension three

Theorem.

If P is a 3-dimensional polytope, then

$$\mathrm{drop}(P) = \begin{cases} 0 & \textit{if } P + (-P) \textit{ is } \underline{\textit{not}} \textit{ a zonotope} \\ 1 & \textit{if } P + (-P) \textit{ is a zonotope, but } P \textit{ itself is } \underline{\textit{not}} \\ 2 & \textit{if } P \textit{ is a zonotope} \end{cases}.$$

We currently have no such characterization in higher dimensions.

McMullen's decomposition

Theorem. (McMullen)

If Ω_0 is translation-invariant, 1-homogeneous and weakly continuous, then there is a valuation ϕ on $(d-1)$-dimensional cones so that

$$\Omega_0(P) = \sum_{e \subset P} \ell_e \, \phi(N_P(e)).$$

Questions:

- How to verify weak continuity?
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$$\Omega_0(P;x) = -\frac{1}{\|x\|^2} \sum_{e \subset P} \ell_e \, \Omega\big(\pi_e T_P(e); \pi_e x\big).$$

McMullen's decomposition for $d = 2$

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For $d = 2$ holds

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Case study: the triangle

$$\frac{-\operatorname{adj}_\Delta}{\langle x, u_1\rangle\langle x, u_2\rangle\langle x, u_3\rangle} = -\frac{1}{\|x\|^2}\Big(\frac{\ell_1}{\langle x, u_1\rangle} + \frac{\ell_2}{\langle x, u_2\rangle} + \frac{\ell_3}{\langle x, u_3\rangle}\Big)$$

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$$\operatorname{adj}_\Delta = \frac{\operatorname{Area}(\Delta)}{\operatorname{CircR}(\Delta)}.$$

McMullen's decomposition for simplices

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$$\Omega_0(P;x) = -\frac{1}{\|x\|^2}\sum_e \ell_e\,\Omega\big(\pi_e T_P(e);\pi_e x\big).$$

First proof idea: triangulate P + prove theorem for simplices.

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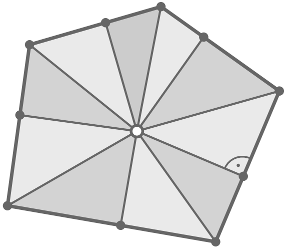
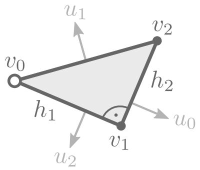
$$\overbrace{\det \begin{pmatrix} | & | & & | \\ u_0 & u_1 & \dots & u_d \\ | & | & & | \\ h_0 & h_1 & \dots & h_d \end{pmatrix}}^{\mathrm{adj}_\Delta = \Omega_0(P;x)\cdot\prod_F \langle x, u_F\rangle} \|x\|^2$$

$$= \sum_{i<j} (-1)^{i+j+d} \underbrace{\det \begin{pmatrix} | & & | & & | & & | \\ u_0 & \cdots & v_i & \cdots & v_j & \cdots & u_d \\ | & & | & & | & & | \\ 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix}}_{\ell_{ij}\, \Omega(T_P(e_{ij}))\cdot \prod_F \langle x, u_F\rangle} \langle u_i, x\rangle \langle u_j, x\rangle.$$

Second proof idea: orthoschemes





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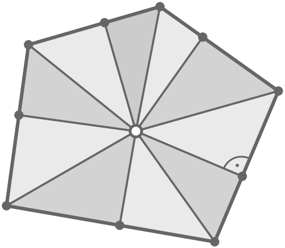
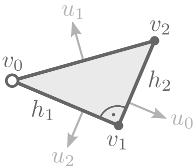
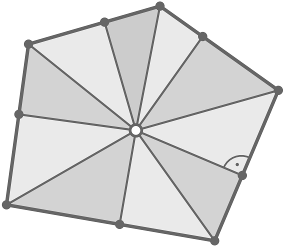
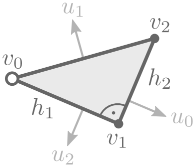




$$
\begin{aligned}
v_0 &= (0, 0, 0, ..., 0), & u_0 &= (h_0, 0, 0, ..., 0, 0),\\
v_1 &= (h_1, 0, 0, ..., 0), & u_1 &= (-h_2, h_1, 0, ..., 0, 0),\\
v_2 &= (h_1, h_2, 0, ..., 0), & u_2 &= (0, -h_3, h_2, ..., 0, 0),\\
v_3 &= (h_1, h_2, h_3, ..., 0), & u_3 &= (0, 0, -h_4, h_3, ..., 0, 0),\\
&\vdots & &\vdots\\
v_d &= (h_1, h_2, h_3, ..., h_d), & u_d &= (0, 0, 0, ..., 0, -h_{d+1}),
\end{aligned}
$$

Second proof idea: orthoschemes





$$\sum_{i=1}^{d} x_i^2 = -\sum_{\substack{i,j=0\\ i<j}}^{d} \frac{h_{i+1}^2 + \cdots + h_j^2}{h_i h_{i+1} h_j h_{j+1}} (h_{i+1} x_i - h_i x_{i+1})(h_{j+1} x_j - h_j x_{j+1}).$$

Thank you.

- $\underline{\mathrm{drop}(P)} := m - d - 1 - \deg \mathrm{adj}_P$ is a well-behaved polytope invariant.
  $\longrightarrow$ **Q:** What else characterizes polytopes of a particular drop?
- Zonotopes are the polytopes with the largest possible drop $d - 1$.
- The <u>reduced canonical form</u> Ω_0 is translation-invariant and 1-homogeneous.
  $\longrightarrow$ Interesting from the perceptive of valuation theory.
  $\longrightarrow$ Useful for questions of scissors congruence.
- Ω_0 gives rise to fascinating polynomial identities involving edge lengths and normal vectors.

"The canonical form, scissors congruence and adjoint degrees of polytopes".
with Tom Baumbach, Ansgar Freyer, Julian Weigert. arXiv:2508.04275