

POLYTOPE RIGIDITY

– GENERIC, CONCRETE AND UNIVERSAL –

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(joint work with Matthias Himmelmann, Bernd Schulze and Zhen “Albert” Zhang)

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES



**combinatorial
synergies**



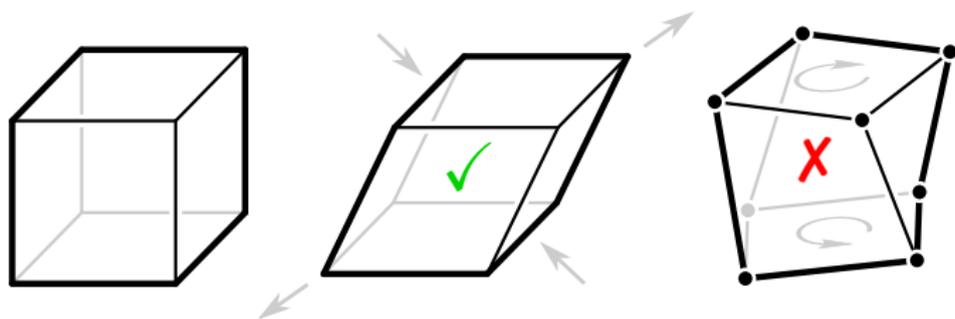
February 25, 2026

POLYTOPE RIGIDITY

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Deforming polytopes in a way that preserves

- ▶ the length of all edges,
- ▶ coplanarity of faces.



Central question

Which polytopes are **rigid**, and which are **flexible**?

POLYTOPE RIGIDITY ... BUT FORMALLY

A **combinatorial type** \mathcal{P} is a triple (V, F, \sim) consisting of

- ▶ a *vertex set* V
- ▶ a *facet set* F
- ▶ a *vertex-facet incidence relation* \sim

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The **realization space** of \mathcal{P} is (we always assume $0 \in \text{int}(P)$)

$$\text{REAL}(\mathcal{P}) := \left\{ \begin{array}{l} \mathbf{p}: V \rightarrow \mathbb{R}^d \\ \mathbf{n}: F \rightarrow \mathbb{R}^d \end{array} \mid \langle \mathbf{p}_i, \mathbf{n}_k \rangle = 1 \text{ if } i \sim k \right\}$$

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A **motion** is a continuous curve $(\mathbf{p}^t, \mathbf{n}^t)$ in $\text{REAL}(\mathcal{P})$ preserving edge lengths:

$$\|\mathbf{p}_i^t - \mathbf{p}_j^t\| \stackrel{!}{=} \ell_{ij} = \text{const} \quad \text{for all } ij \in E$$

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A motion is a **flex** if it is not *trivial* (i.e., just a translation/reorientation).

COUNTING DEGREES OF FREEDOM

$$\#DOFs - \#constraints = d|V| + d|F| - (|E| + |VF|)$$

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$$\# \text{DOFs} - \# \text{constraints} = \underline{3}|V| + \underline{3}|F| - (|E| + |VF|) = \overset{\text{trivial DOFs}}{\downarrow} 6$$

Theorem (LEGENDRE, STEINITZ)

For 3-polytopes, $\text{REALCVX}(\mathcal{P})$ is a smooth semi-algebraic set (i.e., a smooth open manifold) of dimension $|E|$.

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For 3-polytopes, $\text{REALCVX}(\mathcal{P})$ is a smooth semi-algebraic set (i.e., a smooth open manifold) of dimension $|E|$.

- ▶ Imposing one constraint per edge *should* make 3-polytopes rigid.

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For 3-polytopes, $\text{REALCVX}(\mathcal{P})$ is a smooth semi-algebraic set (i.e., a smooth open manifold) of dimension $|E|$.

- ▶ Imposing one constraint per edge *should* make 3-polytopes rigid.
- ▶ In dimension $d \geq 4$ polytopes have many more edges and should be even over-constrained

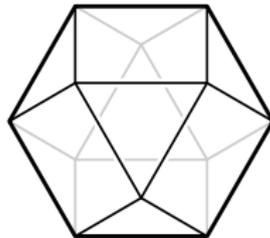
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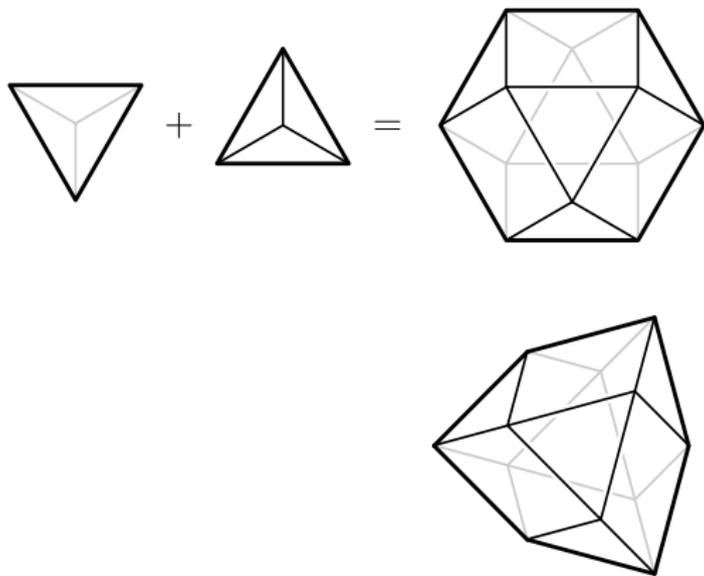
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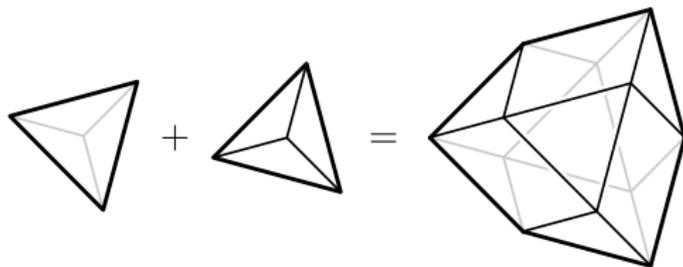
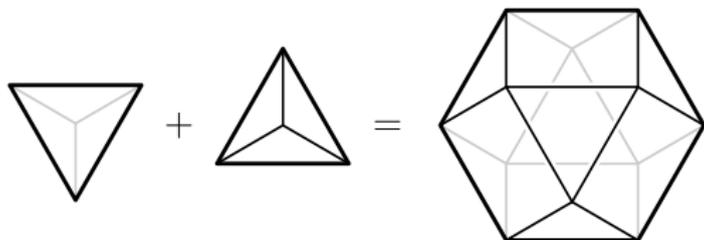
- ▶ Imposing one constraint per edge *should* make 3-polytopes rigid.
- ▶ In dimension $d \geq 4$ polytopes have many more edges and should be even over-constrained ... right?



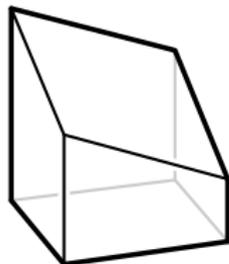
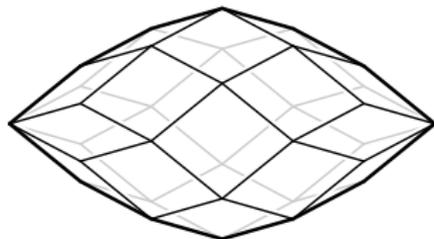
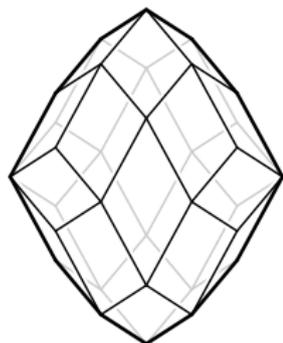
MINKOWSKI FLEXES $A + B := \{a + b \mid a \in A, b \in B\}$



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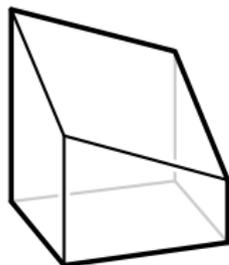
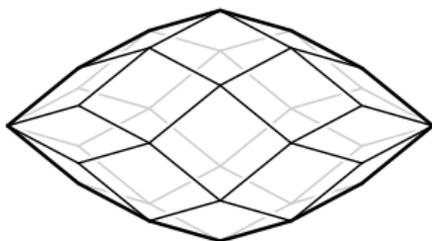
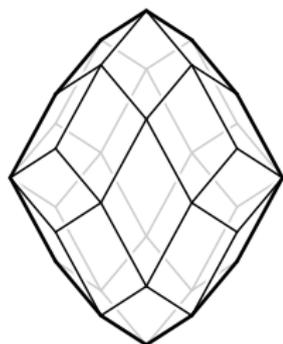
AFFINE FLEXES := a flex realized by an affine transformation



Theorem

- ▶ *A polytope has an affine flex if and only if its edge directions lie on a homogeneous quadric.*
- ▶ *A 3-polytope with at most five edge directions has an affine flex.*

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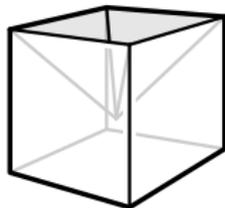
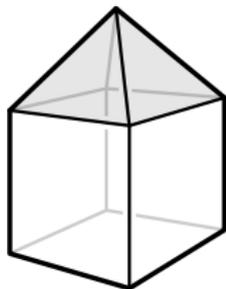
Question.

Is there a polytope flex other than a Minkowski flex?

HISTORY OF POLYTOPE RIGIDITY

Central results

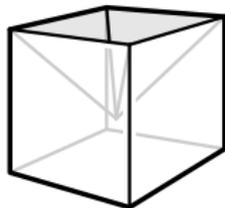
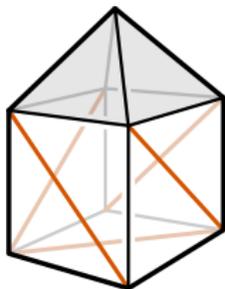
- ▶ *Convex polytopes with rigid 2-faces are (globally) rigid.*
(CAUCHY, ALEXANDROV, 1813)
- ▶ *Convex polytopes with triangulated 2-faces are rigid.*
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- ▶ *Almost all simplicial spheres are (first-order) rigid.* (DEHN, GLUCK)
- ▶ *Flexible simplicial spheres exist.* (CONNELLY, BRICARD, STEFFEN)



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THE STATE OF KNOWLEDGE

- I. *“Rigidity of polytopes with edge length and coplanarity constraints”*
with Matthias Himmelman and Bernd Schulze
see arXiv:2505.00874

“Almost all polytopes are rigid ...”

- II. *“Second-order and global rigidity of polytopes”*
with Matthias Himmelman and Zhen “Albert” Zhang
coming soon

“... but concrete cases are hard to decide ...”

- III. *“Higher-dimensional grid bracing and universality of polytope rigidity”*
with Bernd Schulze
this will take a while

“... and the general case can be universally complicated!”

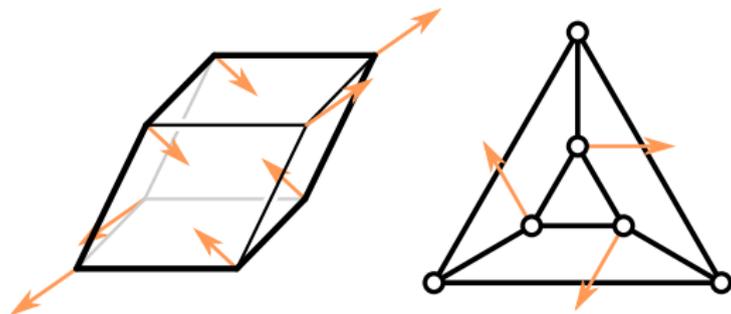
FIRST-ORDER THEORY

FIRST-ORDER MOTIONS

:= deformations that preserves constraints up to first order

A **first-order motion** (\dot{p}, \dot{n}) consists of maps $\dot{p}: V \rightarrow \mathbb{R}^d$ and $\dot{n}: F \rightarrow \mathbb{R}^d$ with

$$\begin{aligned} \|p_i - p_j\|^2 = \text{const} &\xrightarrow{d/dt} \langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 && \text{whenever } ij \in E \\ \langle p_i, n_k \rangle = 1 &\xrightarrow{d/dt} \langle p_i, \dot{n}_k \rangle + \langle \dot{p}_i, n_k \rangle = 0 && \text{whenever } i \sim k \end{aligned}$$



Theorem (ASIMOV, ROTH, 1978)

If a framework is first-order rigid, then it is rigid.

... and one can prove the analogous statement for polytope rigidity.

THE RIGIDITY MATRIX \mathcal{R}_P

$$\begin{array}{c}
 \begin{array}{l}
 \# \text{edges} \\
 \{ \\
 ij \in E \\
 \} \\
 \\
 \# \text{vertex-facet} \\
 \text{incidences} \\
 \{ \\
 i \in F_k \\
 \}
 \end{array}
 \left(
 \begin{array}{c}
 \overbrace{\hspace{10em}}^{d \times \# \text{vertices}} \\
 \begin{array}{cc}
 i \in V & j \in V \\
 \vdots & \vdots \\
 p_i - p_j & p_j - p_i \\
 \vdots & \vdots \\
 \hline
 n_k & p_i \\
 \vdots & \vdots
 \end{array}
 \overbrace{\hspace{10em}}^{d \times \# \text{facets}} \\
 k \in F \\
 \vdots \\
 p_i \\
 \vdots
 \end{array}
 \right)
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$$(\dot{\mathbf{p}}, \dot{\mathbf{n}}) \text{ is a first-order motion} \iff \mathcal{R}_P(\dot{\mathbf{p}}, \dot{\mathbf{n}}) = 0.$$

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 \begin{array}{cc} i \in V & j \in V \\ \vdots & \vdots \end{array} & & k \in F \\
 \hline
 \begin{array}{cc} p_i - p_j & p_j - p_i \\ \vdots & \vdots \end{array} & & \begin{array}{c} \vdots \\ p_i \\ \vdots \end{array}
 \end{array} \right)
 \end{array}$$

$(\dot{\mathbf{p}}, \dot{\mathbf{n}})$ is a first-order motion $\iff \mathcal{R}_P(\dot{\mathbf{p}}, \dot{\mathbf{n}}) = 0$.

$$\# \text{columns} - \# \text{rows} = (3|V| + 3|F|) - (|E| + |VF|) = 6$$

GENERIC RIGIDITY

Theorem (HIMMELMANN, SCHULZE, W., 2025)

A generic realization of a (Zariski) convex 3-polytope is (first-order) rigid.

“Almost all 3-polytopes are rigid.”

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In $d \geq 4$ polytopes *should be over-constrained* by their edges ... but are they?

Conjecture

A generic realization of a d -polytopes ($d \geq 3$) is (first-order) rigid.

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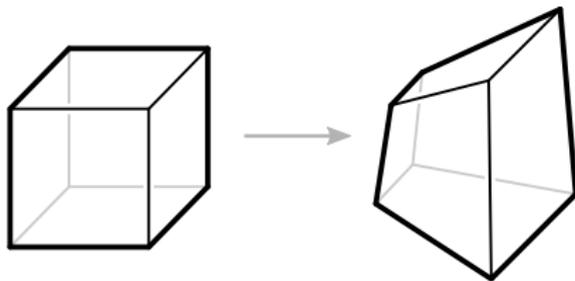
“Almost all 3-polytopes are rigid.”

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A generic *projective transformation* of a d -polytopes ($d \geq 3$) is (first-order) rigid.

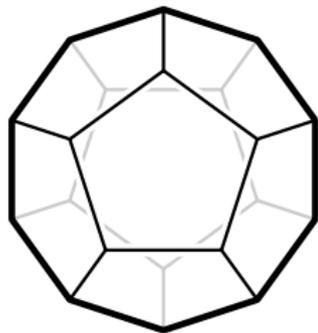
... because maybe polytope flexes need parallel edges.



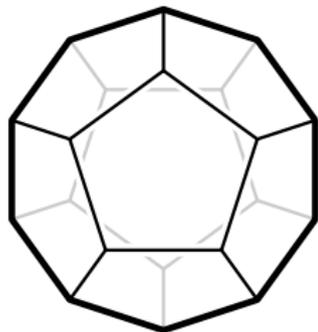
CONCRETE CASES ARE STILL HARD



THE REGULAR DODECAHEDRON



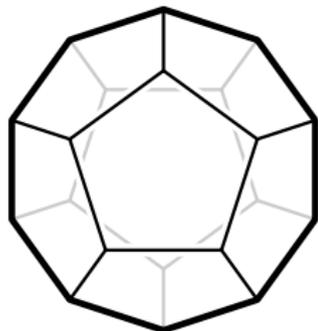
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Facts:

- ▶ 5-dimensional space of first-order flexes!

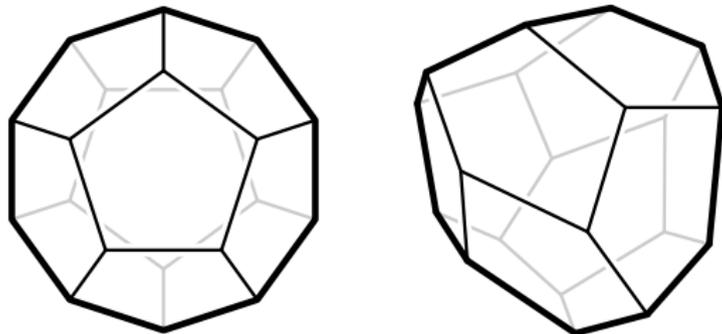
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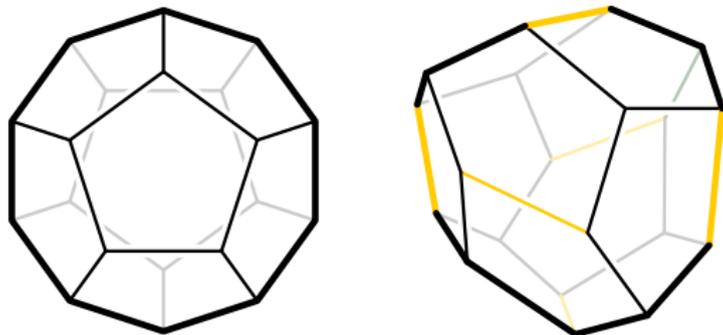
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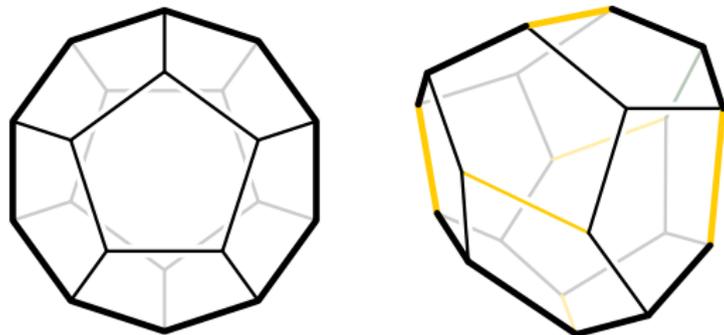
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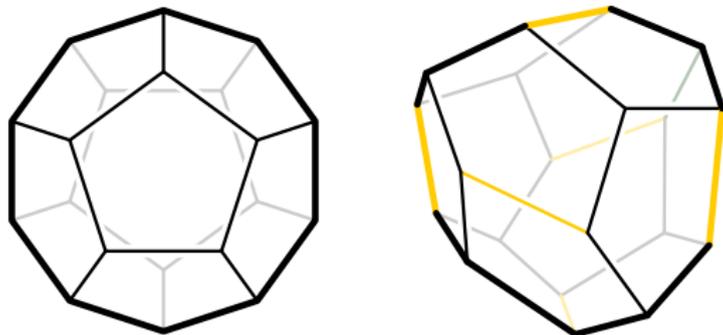
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Theorem (HIMMELMANN, W., ZHANG, 2026+)

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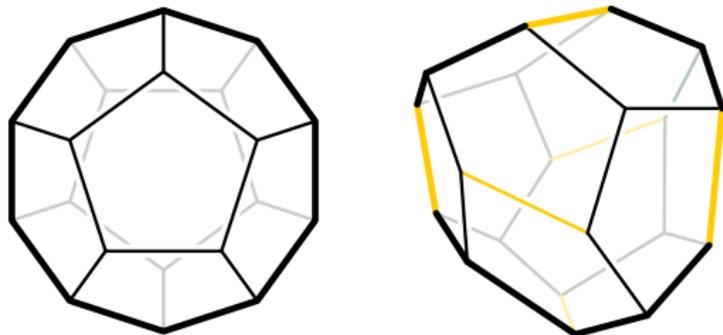
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How hard can this be? It is a small polytope and all constraint are quadratic!

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→ **96** variables + **90** constraints

SECOND-ORDER THEORY

SECOND-ORDER MOTIONS

:= deformations that preserves constraints up to second order.

A **second-order motion** $(\dot{\mathbf{p}}, \ddot{\mathbf{p}}; \dot{\mathbf{n}}, \ddot{\mathbf{n}})$ satisfies

$$\begin{aligned}\langle \dot{p}_i - \dot{p}_j, \dot{p}_i - \dot{p}_j \rangle + \langle p_i - p_j, \ddot{p}_i - \ddot{p}_j \rangle &= 0 && \text{whenever } ij \in E \\ \langle p_i, \ddot{n}_k \rangle + \langle \dot{p}_i, \dot{n}_k \rangle + \langle \ddot{p}_i, n_k \rangle &= 0 && \text{whenever } i \sim k\end{aligned}$$

or equivalently

$$\mathcal{R}_{(\dot{\mathbf{p}}, \dot{\mathbf{n}})}(\dot{\mathbf{p}}, \dot{\mathbf{n}}) + \mathcal{R}_{(p, n)}(\ddot{\mathbf{p}}, \ddot{\mathbf{n}}) = 0.$$

Theorem (CONNELLY, 1996)

If a framework is second-order rigid, then it is rigid.

... and one can prove the analogous statement for polytope rigidity.

CONNELLY'S SECOND-ORDER RIGIDITY TEST

Theorem (*) (CONNELLY, 1996)

A framework is second-order rigid if and only if every first-order flex $\dot{\mathbf{p}}$ is blocked by some stress $\omega : E \rightarrow \mathbb{R}$, i.e.

$$\omega^\top \mathcal{R}_{\dot{\mathbf{p}}} \dot{\mathbf{p}} > 0 \quad \text{or equivalently} \quad \sum_{ij} \omega_{ij} \|\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j\|^2 > 0.$$

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A **stress** is an element of the cokernel of \mathcal{R}_P . For polytopes, stresses consist of $\omega : E \rightarrow \mathbb{R}$ and $\alpha : VF \rightarrow \mathbb{R}$ and satisfy $\mathcal{R}_P^\top(\omega, \alpha) = 0$.

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An analogue to (*) holds for polytope rigidity, where blocking means

$$(\boldsymbol{\omega}, \boldsymbol{\alpha})^\top \mathcal{R}_{(\dot{\mathbf{p}}, \dot{\mathbf{n}})}(\dot{\mathbf{p}}, \dot{\mathbf{n}}) > 0$$

CONNELLY'S SECOND-ORDER RIGIDITY TEST

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A framework is second-order rigid if and only if every first-order flex $\dot{\mathbf{p}}$ is blocked by some stress $\boldsymbol{\omega} : E \rightarrow \mathbb{R}$, i.e.

$$\boldsymbol{\omega}^\top \mathcal{R}_{\dot{\mathbf{p}}} \dot{\mathbf{p}} > 0 \quad \text{or equivalently} \quad \sum_{ij} \omega_{ij} \|\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j\|^2 > 0.$$

A **stress** is an element of the cokernel of \mathcal{R}_P . For polytopes, stresses consist of $\boldsymbol{\omega} : E \rightarrow \mathbb{R}$ and $\boldsymbol{\alpha} : VF \rightarrow \mathbb{R}$ and satisfy $\mathcal{R}_P^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$.

An analogue to (*) holds for polytope rigidity, where blocking means

$$\sum_i \mu_i Q_i(\lambda) = (\boldsymbol{\omega}, \boldsymbol{\alpha})^\top \mathcal{R}_{(\dot{\mathbf{p}}, \dot{\mathbf{n}})}(\dot{\mathbf{p}}, \dot{\mathbf{n}}) > 0$$

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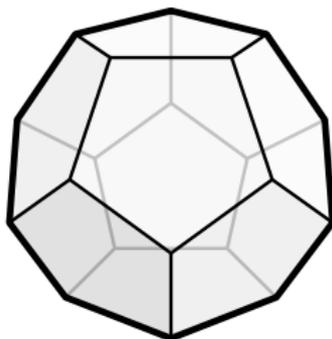
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A polytope is called **prestress stable** if there is a single stress that blocks all first-order flexes.

THE REGULAR DODECAHEDRON



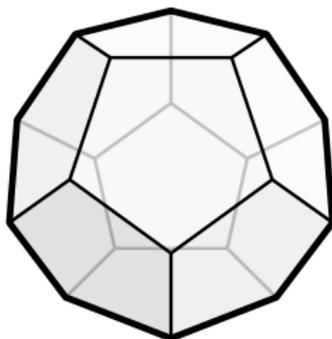
Theorem (HIMMELMANN, W., ZHANG, 2026+)

The regular dodecahedron is ...

- ✗ not first-order rigid. (5-dimensional space of first-order flexes)
- ✗ not prestress stable.
- ✓ second-order rigid.

Note: first example of a natural occurring structure that is second-order rigid but not prestress stable!

THE REGULAR DODECAHEDRON



Theorem (HIMMELMANN, W., ZHANG, 2026+)

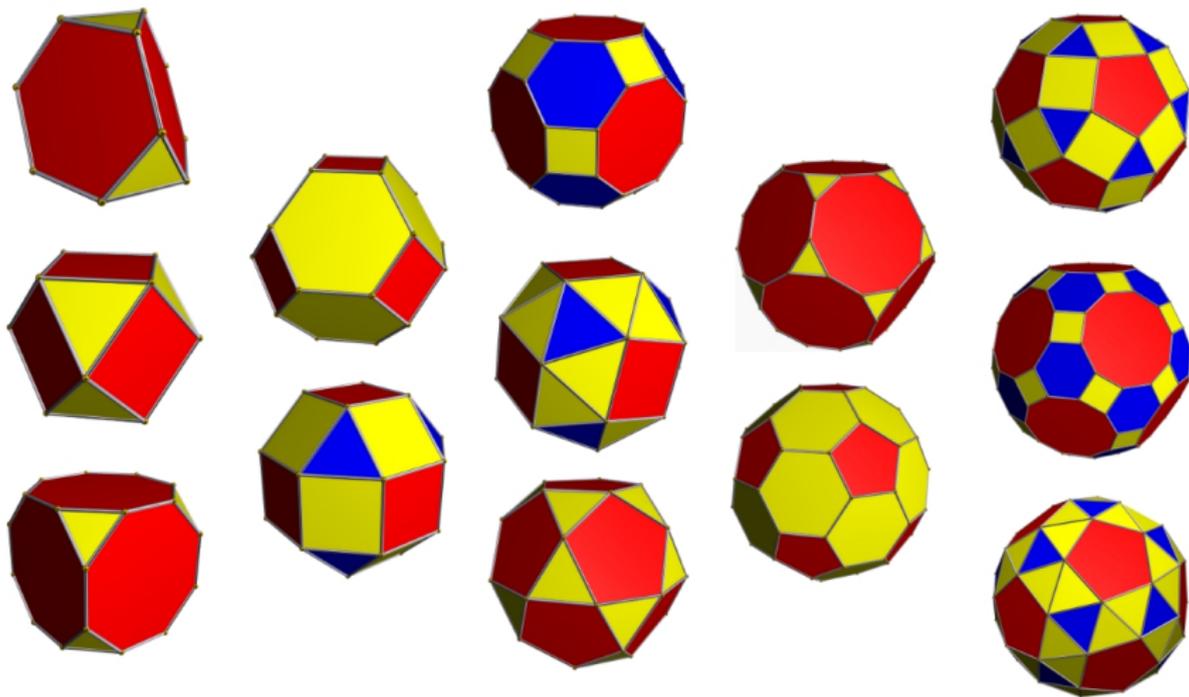
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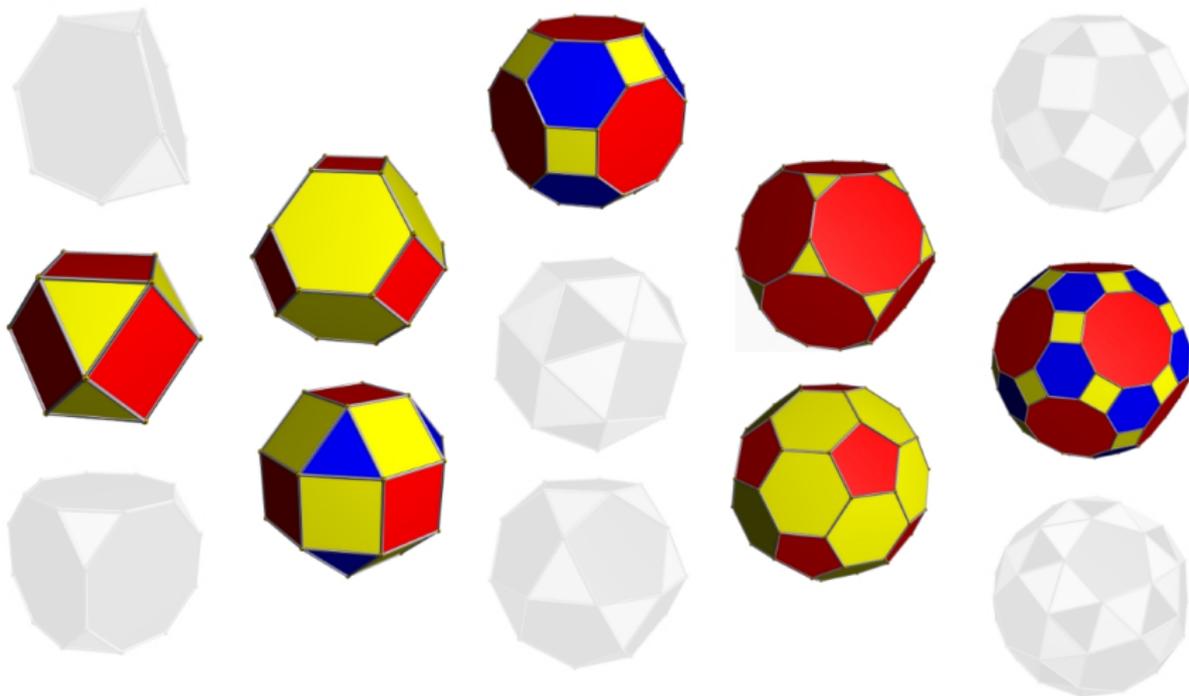
Question

Does rigidity for polyhedra already imply second-order rigidity?

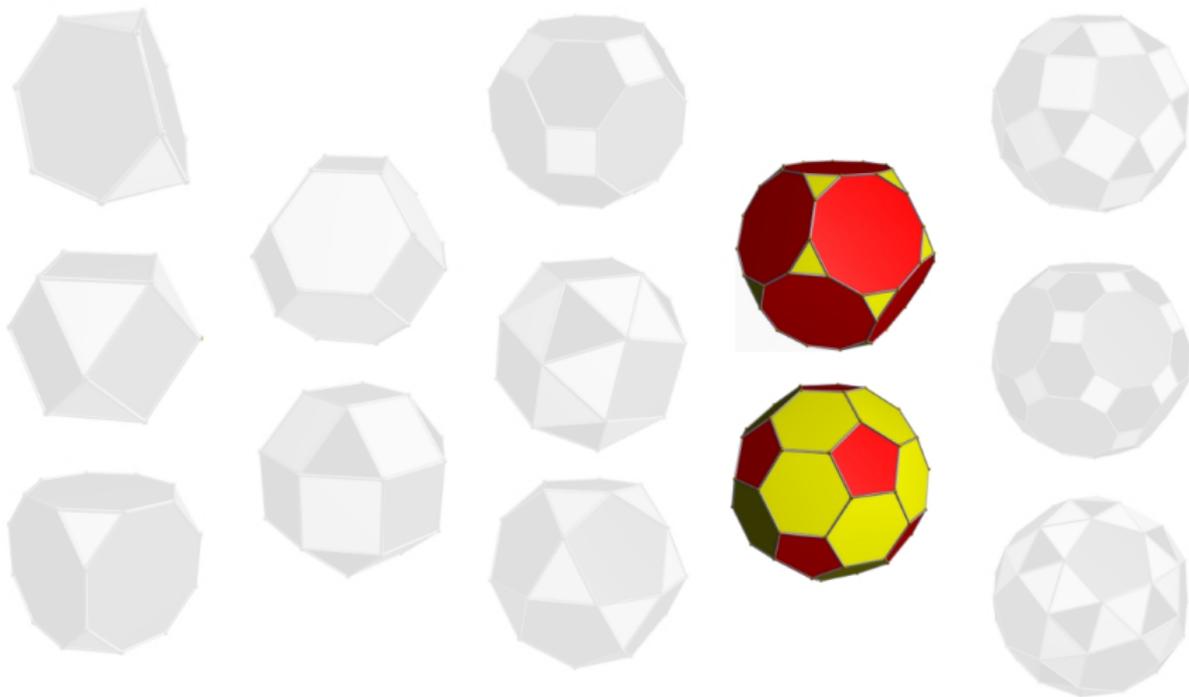
THE ARCHIMEDEAN SOLIDS



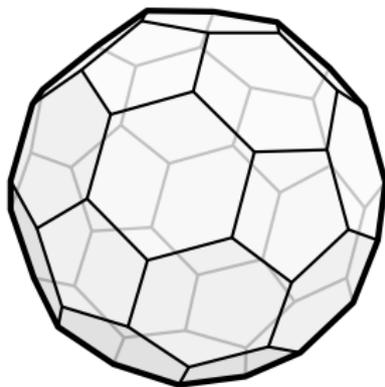
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THE TRUNCATED ICOSAHEDRON



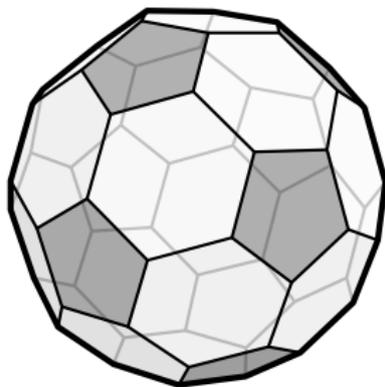
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The truncated icosahedron is ...

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Observation: loses first-order flexes if we truncate on a any other height.

THE TRUNCATED ICOSAHEDRON (THE SOCCER BALL)



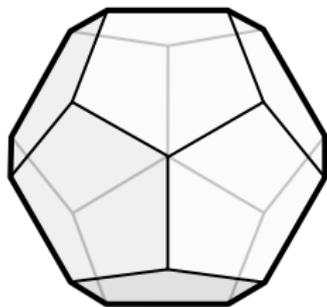
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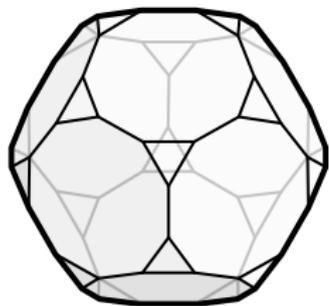
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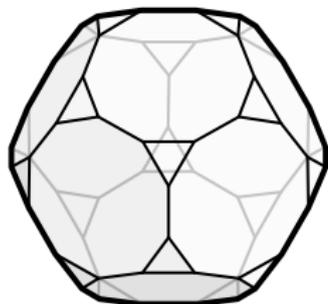
THE TRUNCATED DODECAHEDRON



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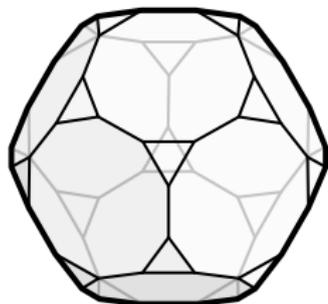


Theorem (HIMMELMANN, W., ZHANG, 2026+)

The truncated dodecahedron is ...

X not first-order rigid. (*4-dimensional space of first-order flexes*)

THE TRUNCATED DODECAHEDRON

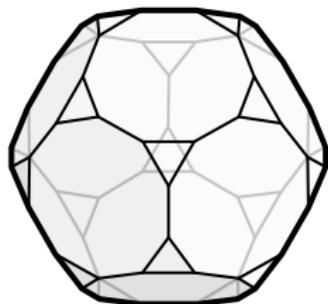


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THE TRUNCATED DODECAHEDRON

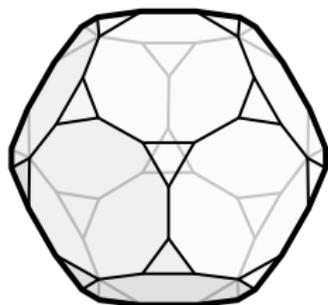


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- ✗ not prestress stable.
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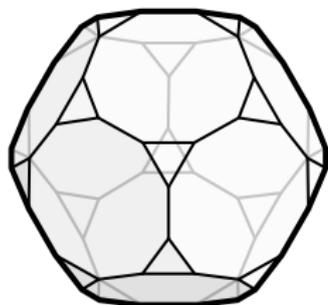
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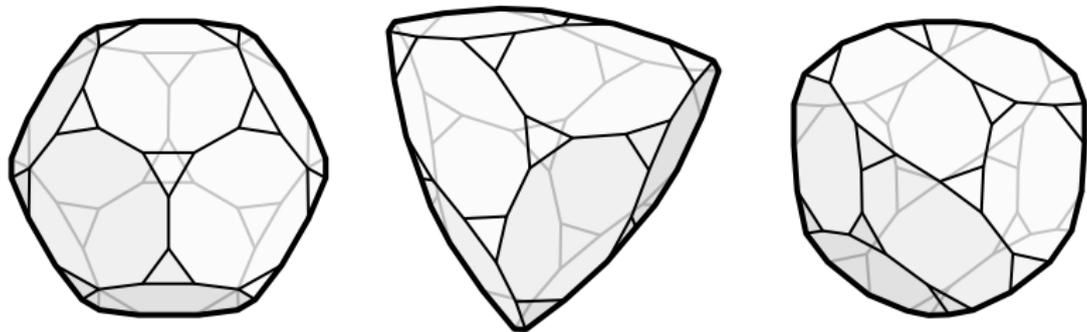
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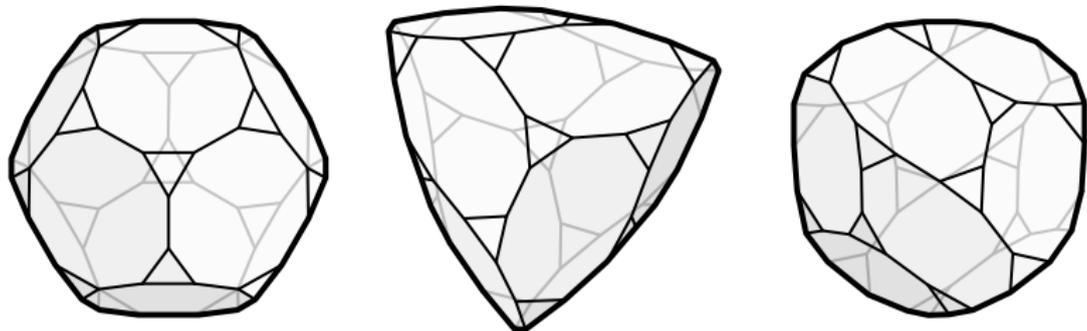
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THE TRUNCATED DODECAHEDRON



Question

Is the truncated dodecahedron rigid?

Either way ...

- ▶ if **yes**: first natural example of a rigid structure that is not second-order rigid.
- ▶ if **no**: first example of a polytope flex in $d \geq 3$ that is not a Minkowski flex.

UNIVERSALITY

“If you can’t show that something is nice, try instead to show that it can become arbitrarily bad!”

UNIVERSALITY THEOREMS

Examples:

- ▶ Mnëv's universality for matroids
- ▶ Richter-Gebert's universality for 4-polytopes (no length constraints)

Kempe's universality theorem (1876)

For every (connected component of an) algebraic curve C there is a linkage with some joint that traces C .

"There is a linkage that can draw your signature."

→ Ultimate confirmation of "behavior too complex for general niceness results".

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“Theorem” (SCHULZE, W., 2026+)

Polytopes of dimension $d \geq 3$ express local universality. That is, for an algebraic set S and point $x \in S$, there is a polytope P whose realization space at P is locally isomorphic to S at x .

STRATEGY

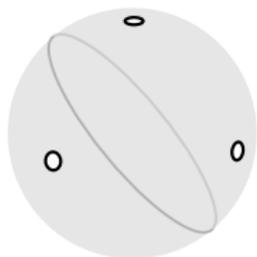
Theorem (KOURGANOFF, 2016)

Kempe's universality theorem holds on the sphere \mathbb{S}^d .

Idea: Given an algebraic set S .

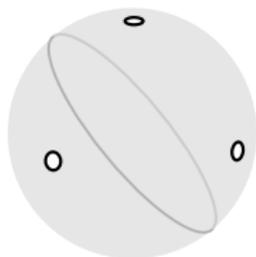
- ▶ choose a spherical Kempe framework (G, \mathbf{p}) whose realization space is (isomorphic to) S .
- ▶ construct a polytope P that simulates the Kempe framework (G, \mathbf{p}) locally.

SIMULATING SPHERICAL FRAMEWORKS



Let (G, \mathbf{p}) be a spherical framework on n vertices:

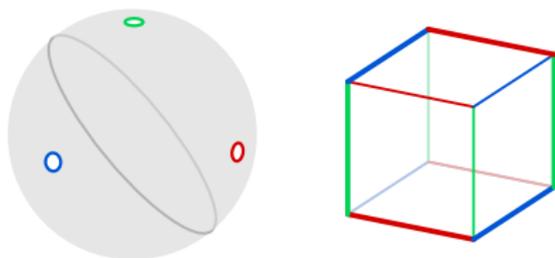
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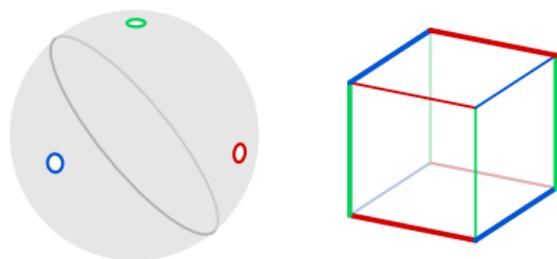
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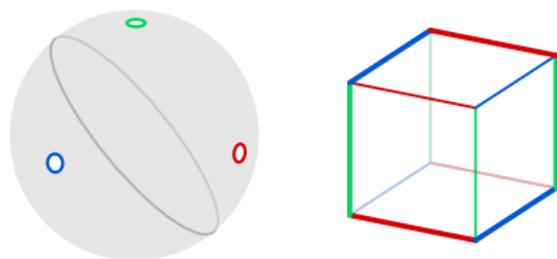
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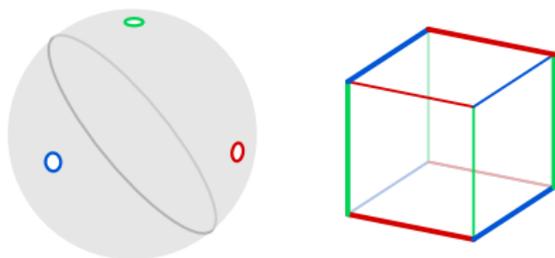
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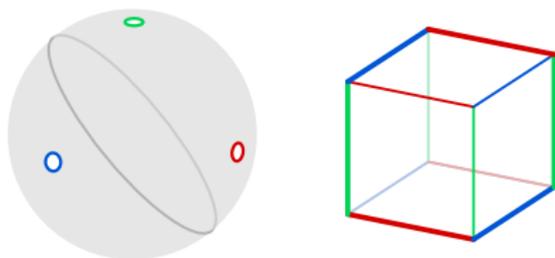
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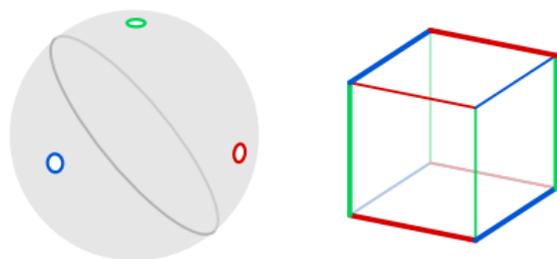
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Let (G, \mathbf{p}) be a spherical framework on n vertices:

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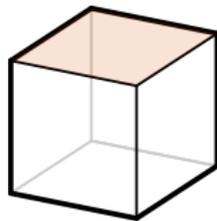
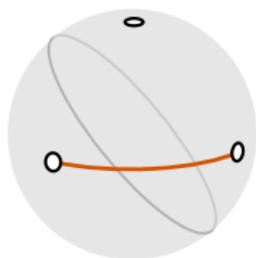
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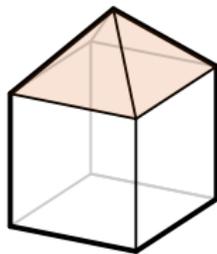
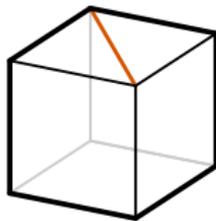
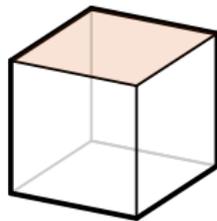
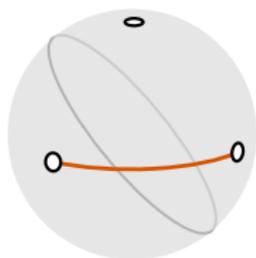
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SIMULATING SPHERICAL FRAMEWORKS

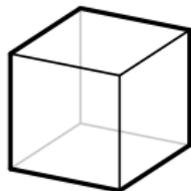


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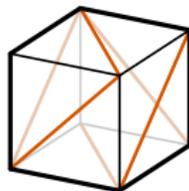
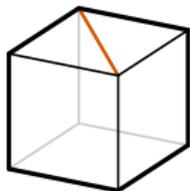
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BRACING ... IS ANNOYING

Bracing = trading a coplanarity constraint by a distance constraint.

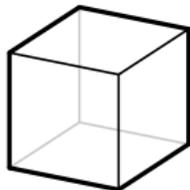


3 DOFs

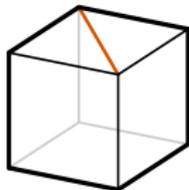


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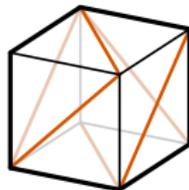
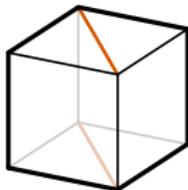
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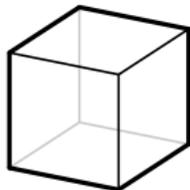


2 DOFs

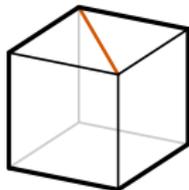


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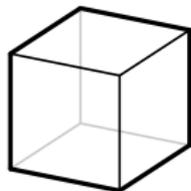


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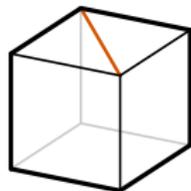


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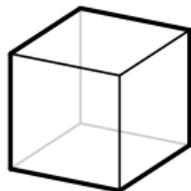


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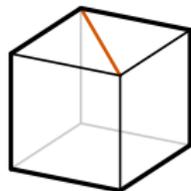


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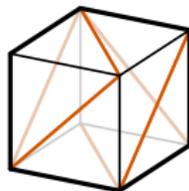
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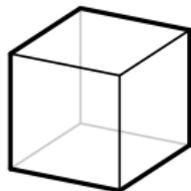


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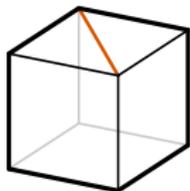


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2 DOFs



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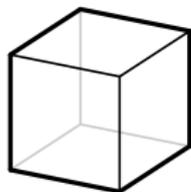
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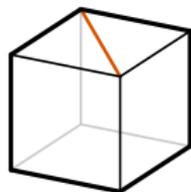
0 DOFs

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Bracing = trading a coplanarity constraint by a distance constraint.



3 DOFs



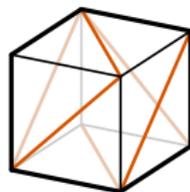
2 DOFs



3 DOFs



3 DOFs

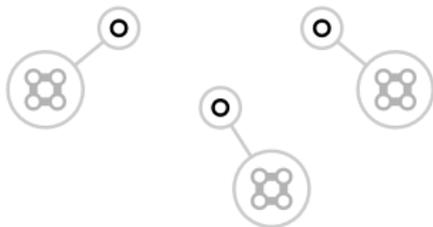
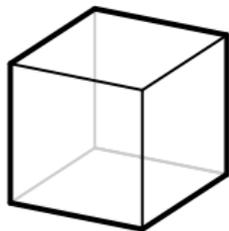


0 DOFs

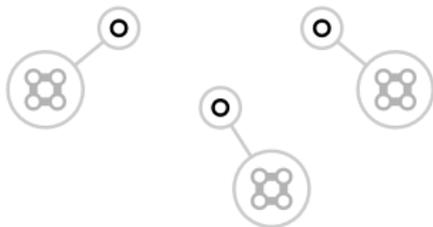
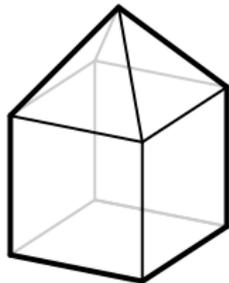
Conclusion:

- ▶ too unpredictable
- ▶ not strictly convex

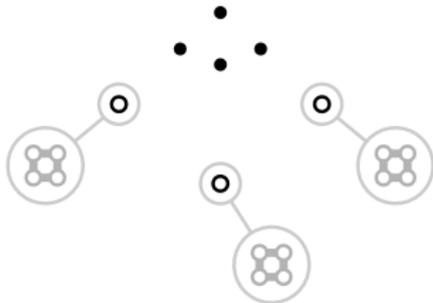
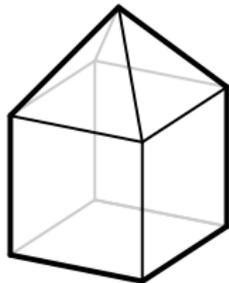
STACKING ... IS BAD TOO, BUT MANAGEABLE



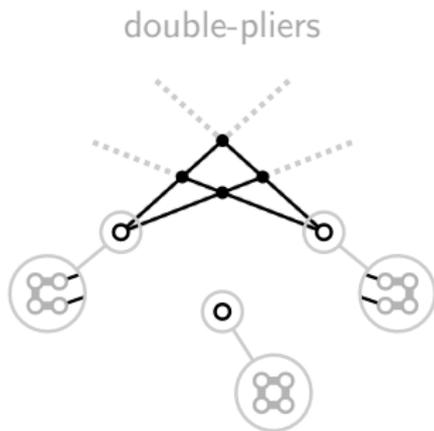
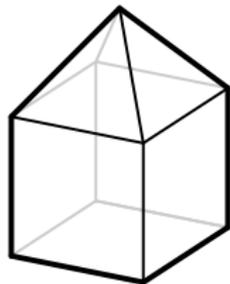
STACKING ... IS BAD TOO, BUT MANAGEABLE



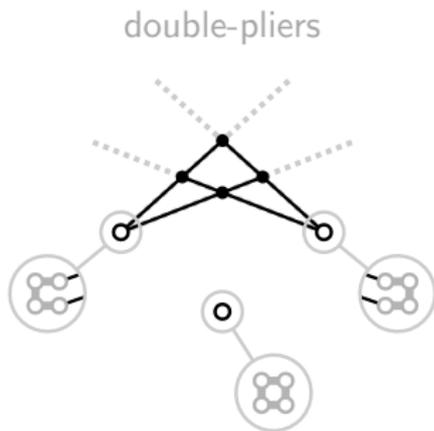
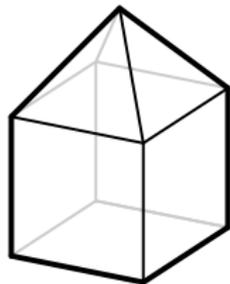
STACKING ... IS BAD TOO, BUT MANAGEABLE



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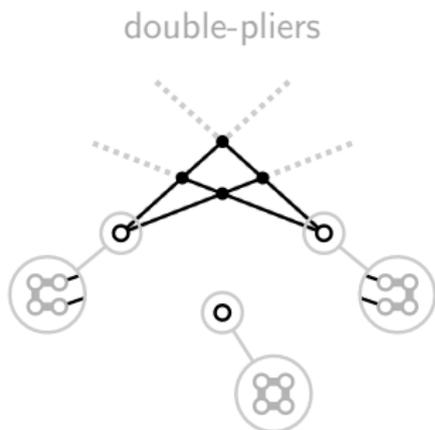
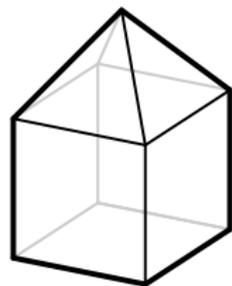
STACKING ... IS BAD TOO, BUT MANAGEABLE



\approx



STACKING ... IS BAD TOO, BUT MANAGEABLE



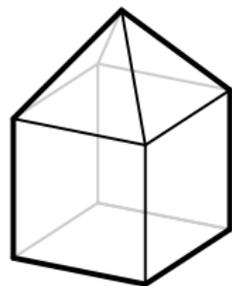
\approx ?



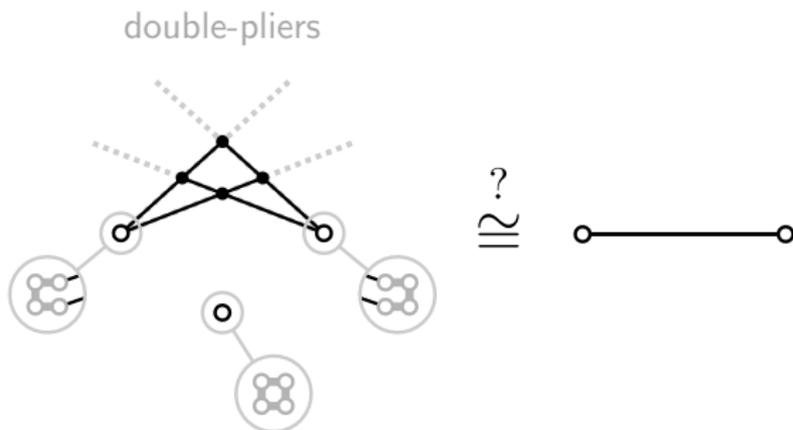
Question: do double-pliers simulate bar constraints? Do they enforce that

- ▶ clusters stay together?
- ▶ clusters stay at constant distance?

STACKING ... IS BAD TOO, BUT MANAGEABLE



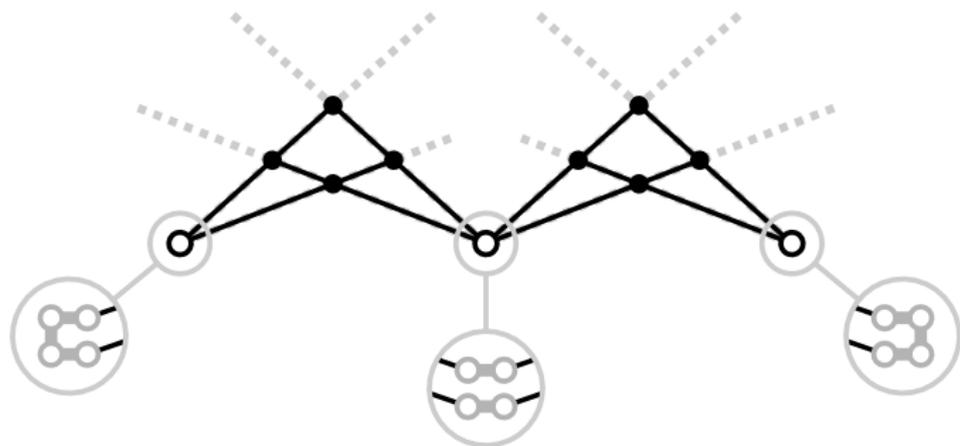
2 DOFs



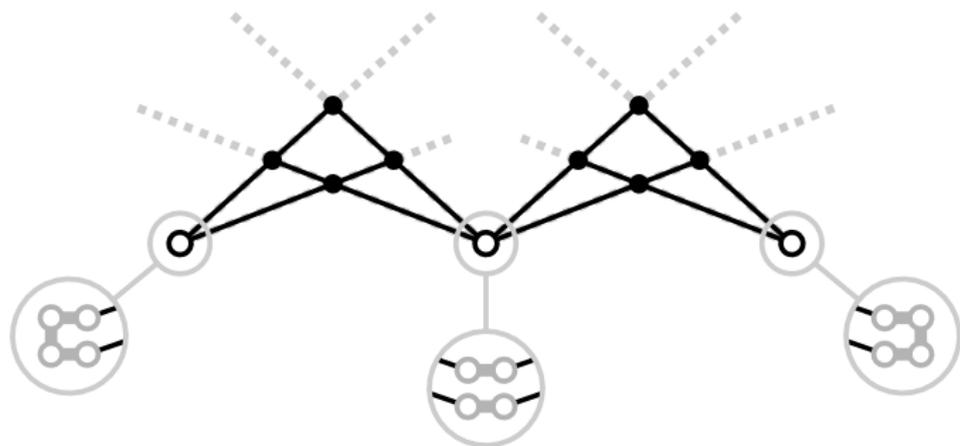
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DEGREE TWO



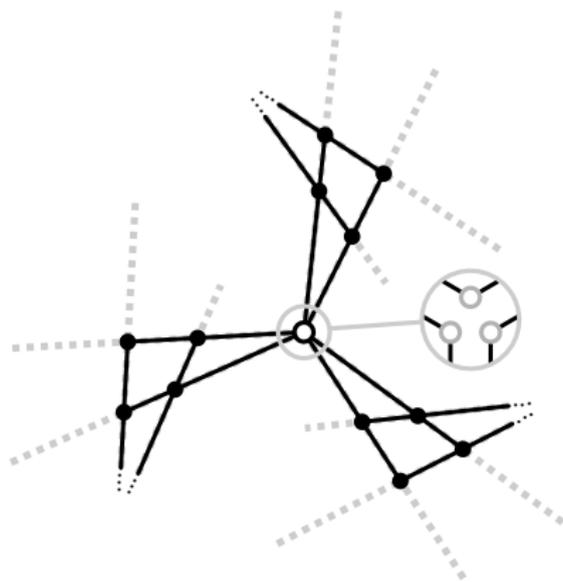
DEGREE TWO



Very technical lemma

Clusters at degree two vertices stay together.

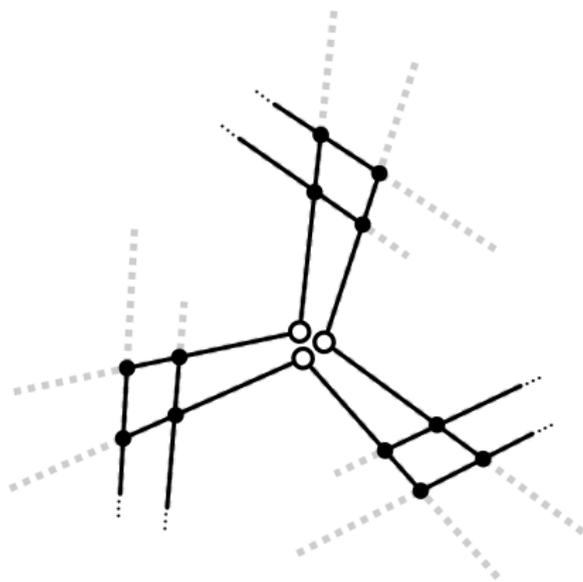
DEGREE THREE



Observations:

- ▶ Clusters at degree-3 vertices can disintegrate. (1 DOFs)

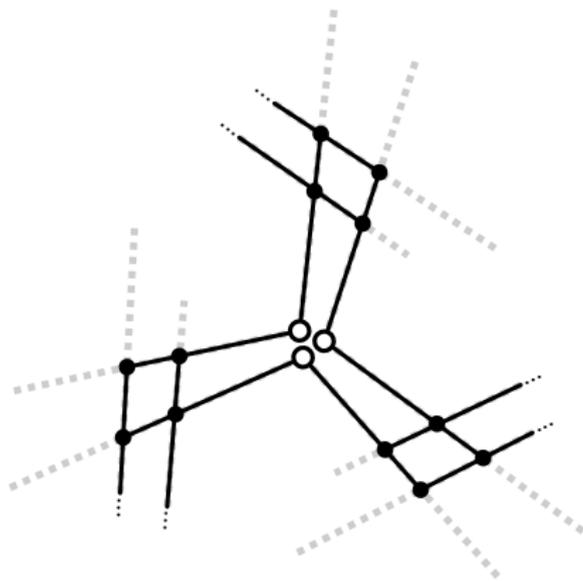
DEGREE THREE



Observations:

- ▶ Clusters at degree-3 vertices can disintegrate. (1 DOFs)

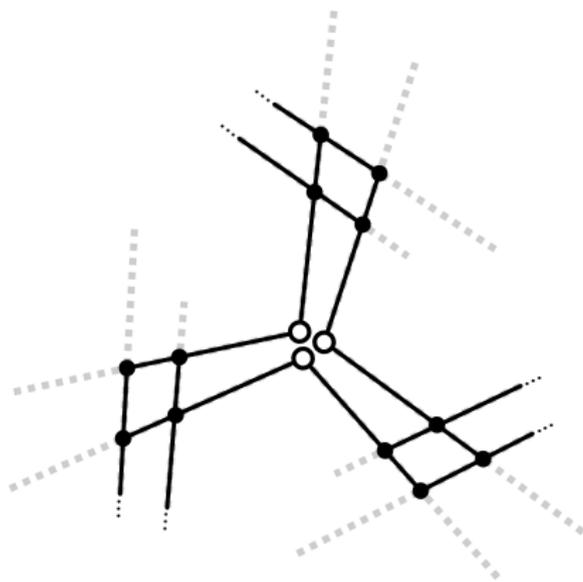
DEGREE THREE



Observations: ($k \geq 3$)

- ▶ Clusters at degree- k vertices can disintegrate. ($k - 2$ DOFs)

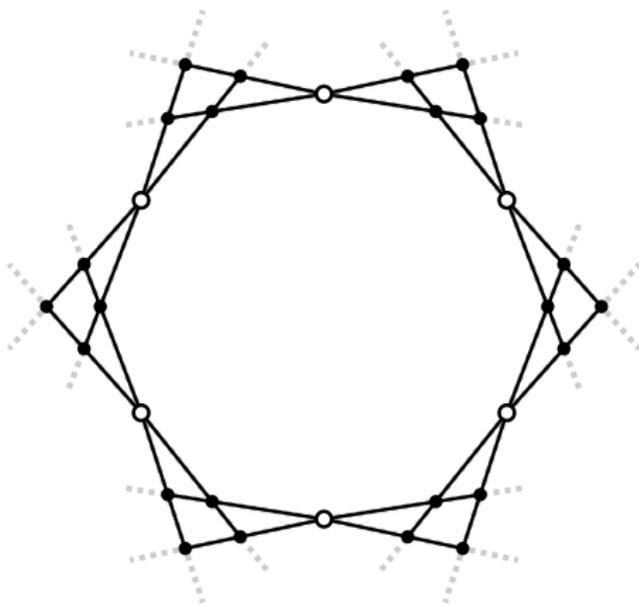
DEGREE THREE



Observations: ($k \geq 3$)

- ▶ Clusters at degree- k vertices can disintegrate. ($k - 2$ DOFs)
- ▶ If one cluster of some double-pliers stays together, then so does the other.
→ In a connected graph, it suffices if a single cluster stays together

MANY MORE TECHNICAL CHALLENGES



- ▶ Cycles impose constraints on the disintegration of clusters
- ▶ Generic choice of pyramid apexes makes these constraints unattainable.

Thank you.

- ▶ Polytope rigidity studies deformations of polytopes that preserve edge lengths and coplanarities
 - ▶ Almost all realizations of a 3-polytope are rigid (true for $d \geq 4$?)
 - ▶ Concrete cases are hard to decide (is the *truncated dodecahedron* rigid?)
 - ▶ In general, local rigidity behavior of polytopes can be universally complicated
- I. *“Rigidity of polytopes with edge length and coplanarity constraints”*
with Matthias Himmelman and Bernd Schulze; arXiv:2505.00874
 - II. *“Second-order and global rigidity of polytopes”*
with Matthias Himmelman and Zhen “Albert” Zhang (coming soon)
 - III. *“Higher-dimensional grid bracing and universality of polytope rigidity”*
with Bernd Schulze