Polytope Rigidity
– Generic, concrete and universally bad –

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(joint work with Matthias Adrian-Himmelmann, Bernd Schulze and Albert Zhang)

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

combinatorial synergies

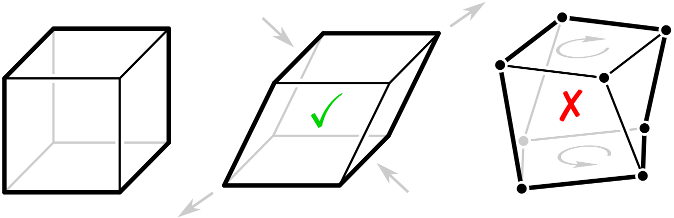
May 18, 2026

Polytope Rigidity

Polytope rigidity

Deforming polytopes in a way that preserves

- the length of all edges,
- coplanarity of faces.



Central question

Which polytopes are **rigid**, and which are **flexible**?

Polytope rigidity ... but formally

A **combinatorial type** $\mathcal{P}$ is a triple $(V, F, \sim)$ consisting of

- a *vertex set* V
- a *facet set* F
- a *vertex-facet incidence relation* $\sim$

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The **realization space** of $\mathcal{P}$ is (we always assume $0 \in \mathrm{int}(P)$)

$$\textsc{Real}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^d \\ \boldsymbol{n}\colon F \to \mathbb{R}^d \end{array} \;\middle|\; \langle p_i, n_k \rangle = 1 \;\text{ if } i \sim k \right\}$$

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$$\textsc{RealCvx}(\mathcal{P}) := \left\{ \begin{array}{l} \boldsymbol{p}\colon V \to \mathbb{R}^d \\ \boldsymbol{n}\colon F \to \mathbb{R}^d \end{array} \;\middle|\; \begin{array}{l} \langle p_i, n_k \rangle = 1 \text{ if } i \sim k \\ \langle p_i, n_k \rangle < 1 \text{ if } i \not\sim k \end{array} \right\}$$

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A **motion** is a continuous curve $(\boldsymbol{p}^t, \boldsymbol{n}^t)$ in $\mathrm{R\small{EAL}}(\mathcal{P})$ preserving edge lengths:

$$\|p_i^t - p_j^t\| \overset{!}{=} \ell_{ij} = \text{const} \qquad \text{for all } ij \in E$$

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A motion is a **flex** if it is not *trivial* (i.e., just a translation/reorientation).

Counting degrees of freedom

$$\#\text{DOFs} - \#\text{constraints} = d|V| + d|F| - \big(|E| + |VF|\big)$$

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Theorem (Legendre, Steinitz)

For 3-polytopes, $\text{RealCvx}(\mathcal{P})$ is a smooth semi-algebraic set (i.e., a smooth open manifold) of dimension $|E| + 6$.

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- Imposing one constraint per edge *should* make 3-polytopes rigid.

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- In dimension $d \geq 4$ polytopes have many more edges and should be even over-constrained

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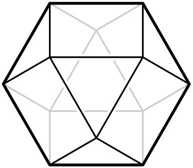
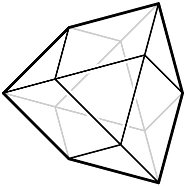
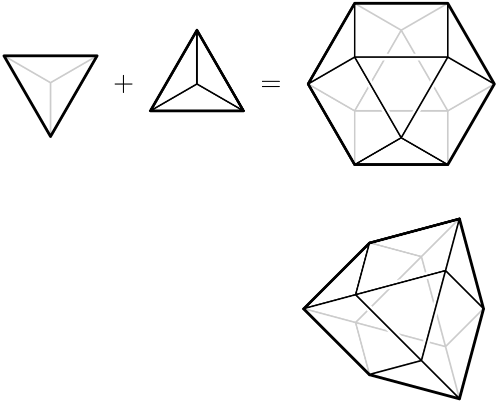
- Imposing one constraint per edge *should* make 3-polytopes rigid.
- In dimension $d \geq 4$ polytopes have many more edges and should be even over-constrained ... right?





Minkowski flexes $A + B := \{a + b \mid a \in A, b \in B\}$



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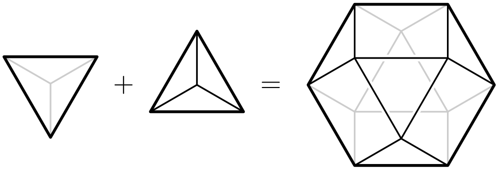
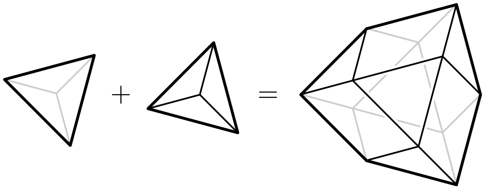
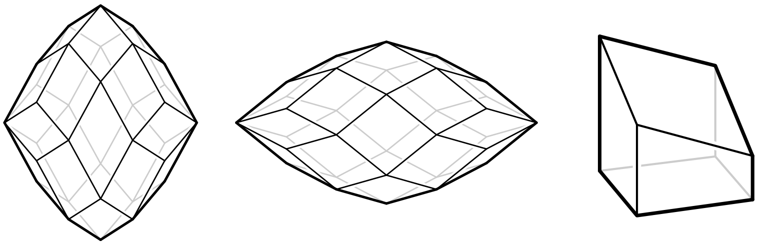




AFFINE FLEXES := a flex realized by an affine transformation



Theorem

- *A polytope has an affine flex if and only if its edge directions lie on a homogeneous quadric.*
- *A 3-polytope with at most five edge directions has an affine flex.*

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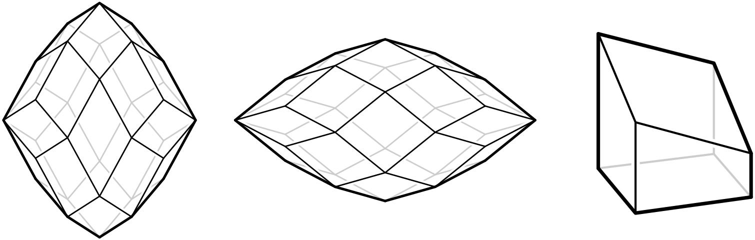

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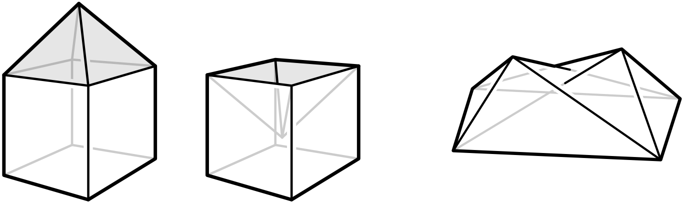
Question.

Is there a polytope flex other than a Minkowski flex?

History of polytope rigidity

Central results

- *Convex polytopes with rigid 2-faces are (gloabally) rigid.*
  (Cauchy, Alexandrov, *1813)*
- *Convex polytopes with triangulated 2-faces are rigid.*
  (Alexandrov, Whiteley, Connelly, Gortler)
- *Almost all simplicial spheres are (first-order) rigid.* (Dehn, Gluck)
- *Flexible simplicial spheres exist.* (Connelly, Bricard, Steffen)



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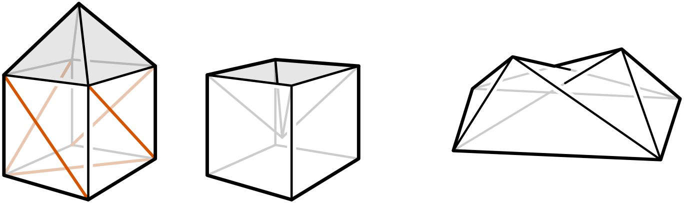

The state of knowledge

I. *"Rigidity of polytopes with edge length and coplanarity constraints"*
with Matthias Himmelmann and Bernd Schulze
see arXiv:2505.00874

"Almost all polytopes are rigid ..."

II. *"Second-order and global rigidity of polytopes"*
with Matthias Himmelmann and Zhen "Albert" Zhang
coming soon

"... but concrete cases are hard to decide ..."

III. *"Higher-dimensional grid bracing and universality of polytope rigidity"*
with Bernd Schulze
this will take a while

"... and the general case can be universally complicated!"

First-order theory

First-order motions

:= deformations that preserves constraints up to first order

A **first-order motion** $(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})$ consists of maps $\dot{\boldsymbol{p}}\colon V \to \mathbb{R}^d$ and $\dot{\boldsymbol{n}}\colon F \to \mathbb{R}^d$ with

$$\|p_i - p_j\|^2 = \text{const} \quad \overset{\mathrm{d}/\mathrm{d}t}{\Longrightarrow} \quad \langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad \text{whenever } ij \in E$$

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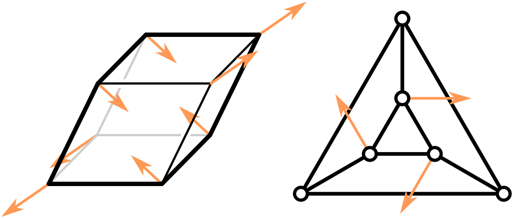
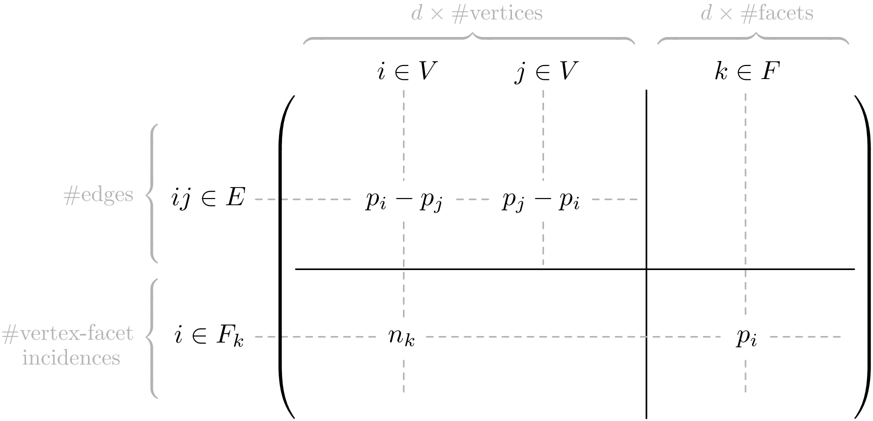


Theorem (Asimov, Roth, 1978)

If a framework is first-order rigid, then it is rigid.

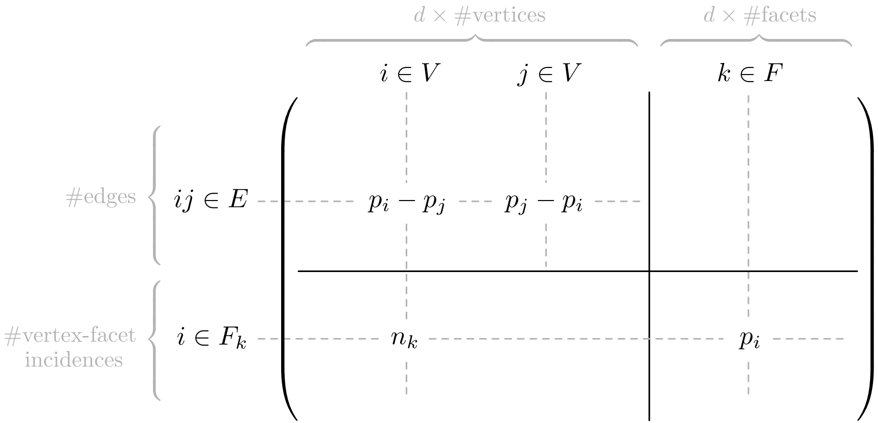
... and one can prove the analogous statement for polytope rigidity.

THE RIGIDITY MATRIX $\mathcal{R}_P$



$(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})$ is a first-order motion $\iff \mathcal{R}_P(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}}) = 0.$

The rigidity matrix $\mathcal{R}_P$



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Theorem (Himmelmann, Schulze, W., 2025)

A generic realization of a (Zariski) convex 3-polytope is (first-order) rigid.

"Almost all 3-polytopes are rigid."

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In $d \geq 4$ polytopes *should be over-constrained* by their edges ... but are they?

Conjecture

A generic realization of a d-polytopes ($d \geq 3$) is (first-order) rigid.

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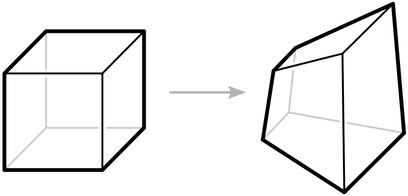
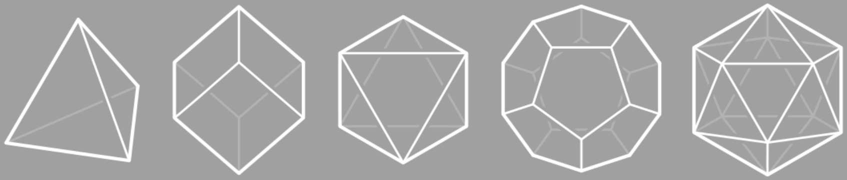
A generic projective transformation of a d-polytopes ($d \geq 3$) is (first-order) rigid.

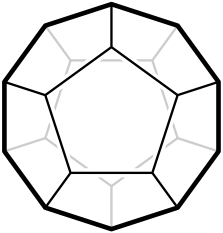
... because maybe polytope flexes need parallel edges.



Concrete cases are still hard



The regular dodecahedron



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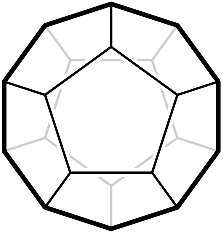

Facts:

- 5-dimensional space of first-order flexes!

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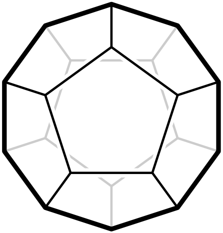

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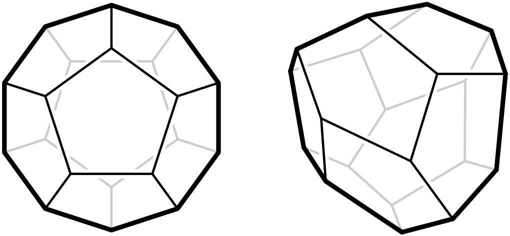

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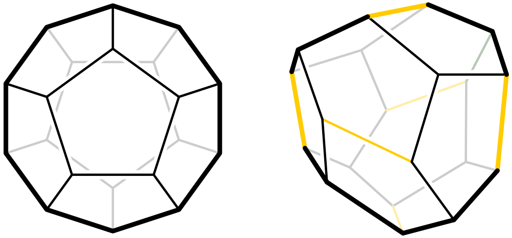

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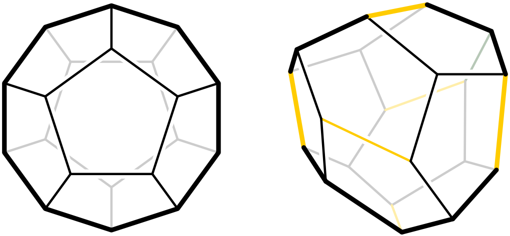

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Theorem (Himmelmann, W., Zhang, 2026+)

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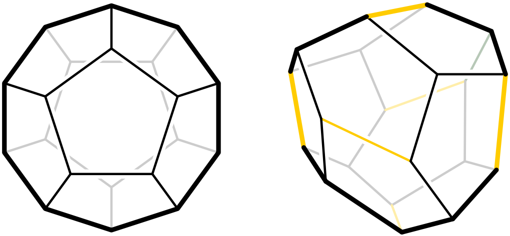

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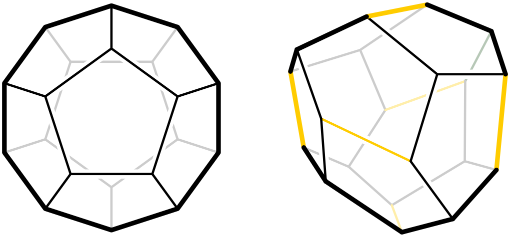

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$\longrightarrow$ **96** variables + **90** constraints

Second-order theory

Second-order motions

:= deformations that preserves constraints up to second order.

A **second-order motion** $(\dot{\boldsymbol{p}}, \ddot{\boldsymbol{p}}; \dot{\boldsymbol{n}}, \ddot{\boldsymbol{n}})$ satisfies

$$\begin{aligned} \langle \dot p_i - \dot p_j, \dot p_i - \dot p_j \rangle + \langle p_i - p_j, \ddot p_i - \ddot p_j \rangle &= 0 \qquad \text{whenever } ij \in E \\ \langle p_i, \ddot n_k \rangle + \langle \dot p_i, \dot n_k \rangle + \langle \ddot p_i, n_k \rangle &= 0 \qquad \text{whenever } i \sim k \end{aligned}$$

or equivalently

$$\mathcal{R}_{(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})}(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}}) + \mathcal{R}_{(\boldsymbol{p}, \boldsymbol{n})}(\ddot{\boldsymbol{p}}, \ddot{\boldsymbol{n}}) = 0.$$

Theorem (Connelly, Whiteley, 1996)

If a framework is second-order rigid, then it is rigid.

... and one can prove the analogous statement for polytope rigidity.

Second-order rigidity test

Theorem $(*)$ (Connelly, Whiteley, 1996)

A framework is second-order rigid if and only if every first-order flex $\dot{\boldsymbol{p}}$ *is blocked by some stress* $\boldsymbol{\omega}: E \to \mathbb{R}$, *i.e.*

$$\boldsymbol{\omega}^\top \mathcal{R}_{\dot{\boldsymbol{p}}} \dot{\boldsymbol{p}} > 0 \qquad \text{or equivalently} \qquad \sum_{ij} \omega_{ij} \|\dot{p}_i - \dot{p}_j\|^2 > 0.$$

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A **stress** is an element of the cokernel of $\mathcal{R}_P$. For polytopes, stresses consist of $\boldsymbol{\omega}: E \to \mathbb{R}$ and $\boldsymbol{\alpha}: VF \to \mathbb{R}$ and satisfy $\mathcal{R}_P^\top(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$.

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An analogue to $(*)$ holds for polytope rigidity, where blocking means

$$(\boldsymbol{\omega}, \boldsymbol{\alpha})^\top \mathcal{R}_{(\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}})} (\dot{\boldsymbol{p}}, \dot{\boldsymbol{n}}) > 0$$

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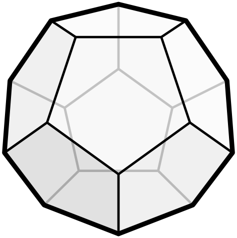
A polytope is called **prestress stable** if there is a single stress that blocks all first-order flexes.

The regular dodecahedron



Theorem (Himmelmann, W., Zhang, 2026+)

The regular dodecahedron is ...

- ✗ *not first-order rigid.* (*5-dimensional space of first-order flexes*)
- ✗ *not prestress stable.*
- ✓ *second-order rigid.*

Note: first example of a natural occurring structure that is second-order rigid but not prestress stable!

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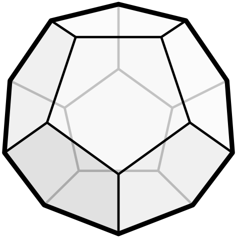

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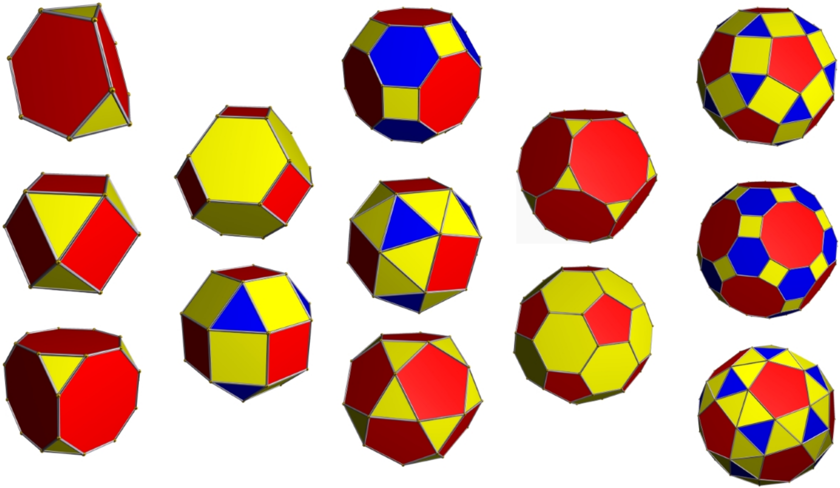
Question

Does rigidity for polyhedra already imply second-order rigidity?

The Archimedean solids



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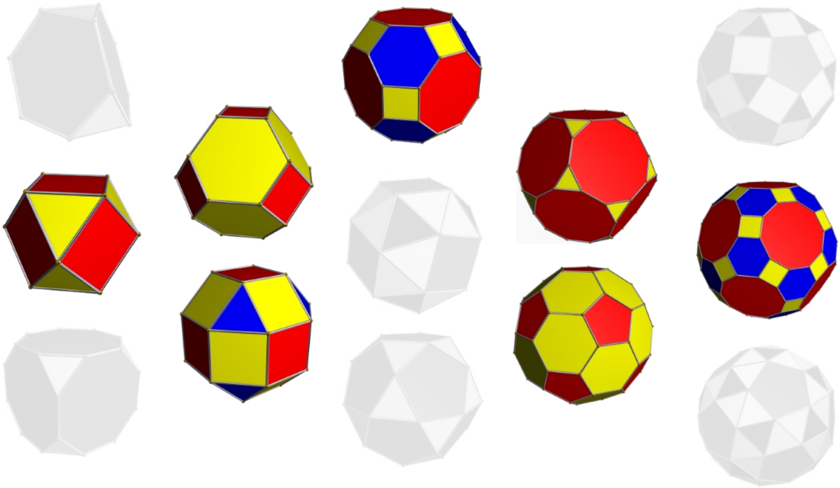

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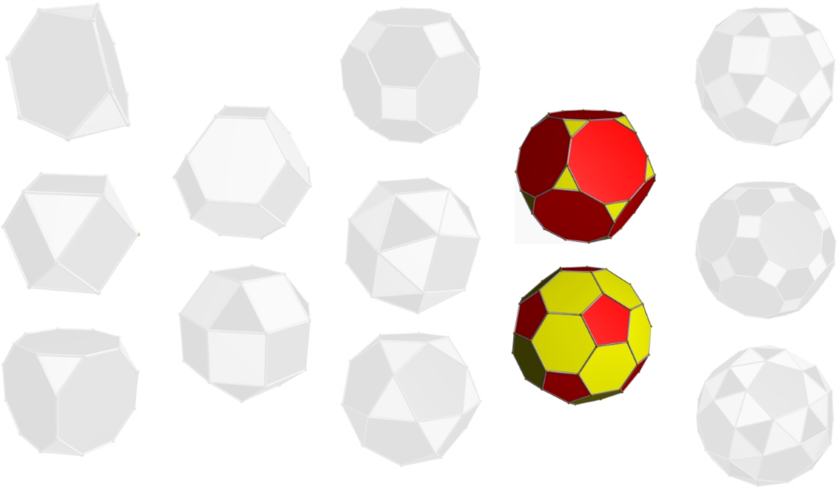
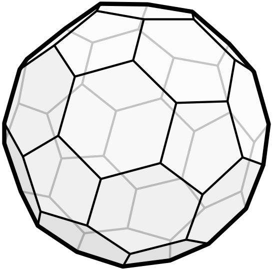


The truncated icosahedron



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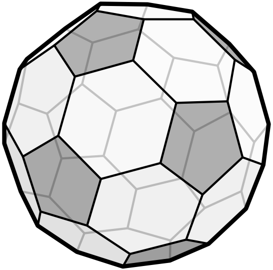
Observation: loses first-order flexes if we truncate on a any other height.

The truncated icosahedron (the soccer ball)



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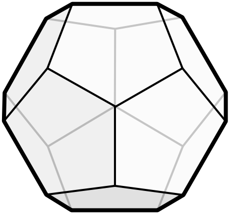

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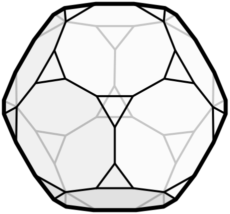

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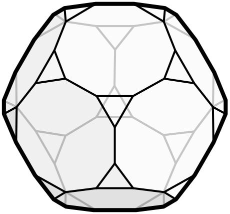

Theorem (Himmelmann, W., Zhang, 2026+)

The truncated dodecahedron is ...

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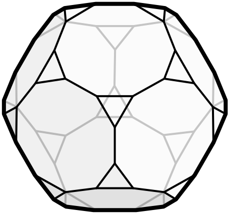

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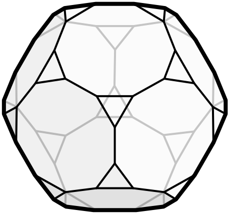

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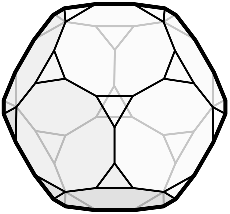

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- ? *rigid* ? ⟶ *there is no third-order rigidity test!*

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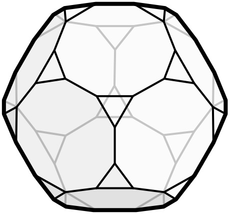

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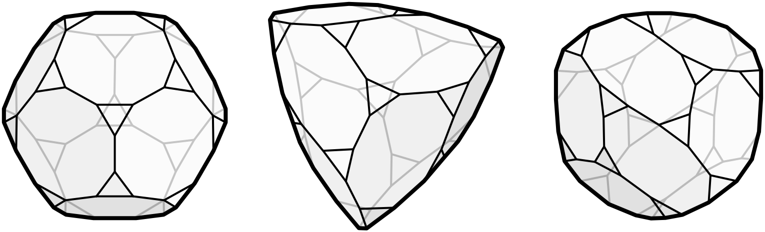

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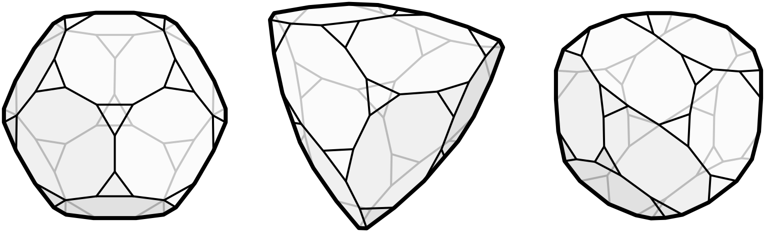

Question

Is the truncated dodecahedron rigid?

Either way ...

- if **yes**: first natural example of a rigid structure that is not second-order rigid.
- if **no**: first example of a polytope flex in $d \geq 3$ that is not a Minkowski flex.

Universality

"If you can't show that something is nice, try instead to show that it can become arbitrarily bad!"

Universality theorems

Examples:

- Mnëv's universality for matroids
- Richter-Gebert's universality for 4-polytopes (no length constraints)

Kempe's universality theorem (1876)

For every (connected component of an) algebraic curve C there is a linkage with some joint that traces C.

"There is a linkage that can draw your signature."

$\longrightarrow$ Ultimate confirmation of "behavior too complex for general niceness results".

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"Theorem" (Schulze, W., 2026+)

Polytopes of dimension $d \geq 3$ express local universality. That is, for an algebraic set S and point $x \in S$, there is a polytope P whose realization space at P is locally isomorphic to S at x.

Strategy

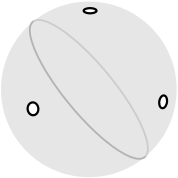
Theorem (Kourganoff, 2016)

Kempe's universality theorem holds on the sphere $\mathbb{S}^d$.

Idea: Given an algebraic set S.

- choose a spherical Kempe framework $(G, \boldsymbol{p})$ whose realization space is (isomorphic to) S.
- construct a polytope P that simulates the Kempe framework $(G, \boldsymbol{p})$ locally.

Simulating spherical frameworks



Let $(G, \boldsymbol{p})$ be a spherical framework on n vertices:

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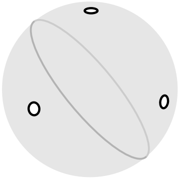

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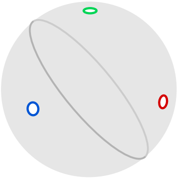

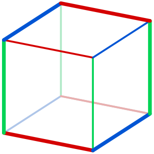

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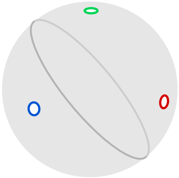

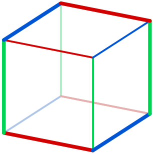

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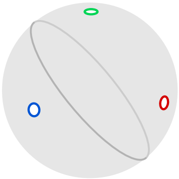

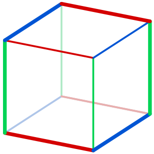

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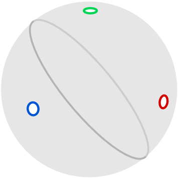

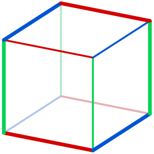

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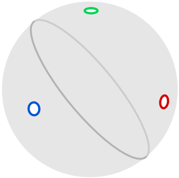

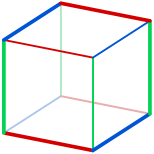

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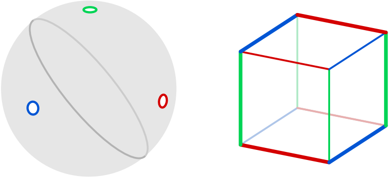

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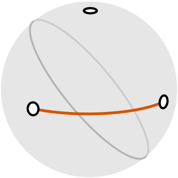

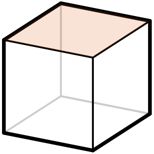

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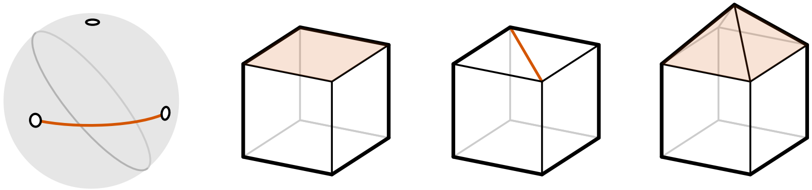
Simulating spherical frameworks



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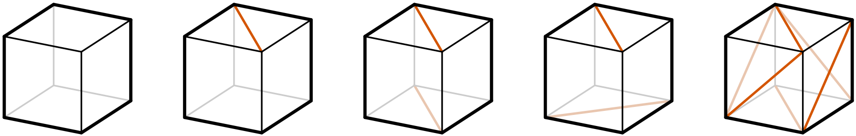
Bracing ... is annoying

Bracing = trading a coplanarity constraint by a distance constraint.



3 DOFs

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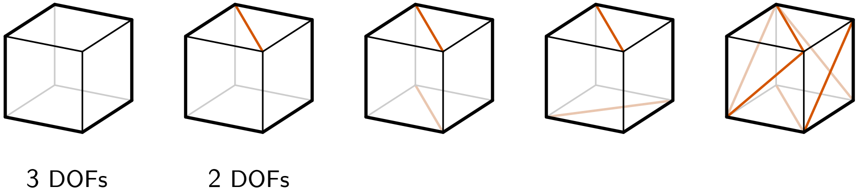

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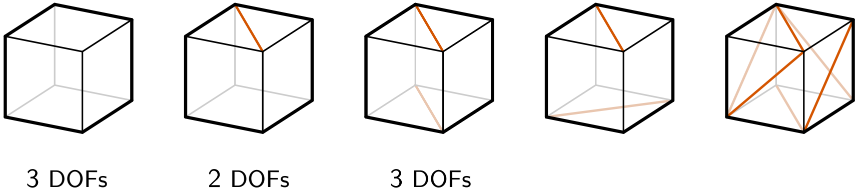

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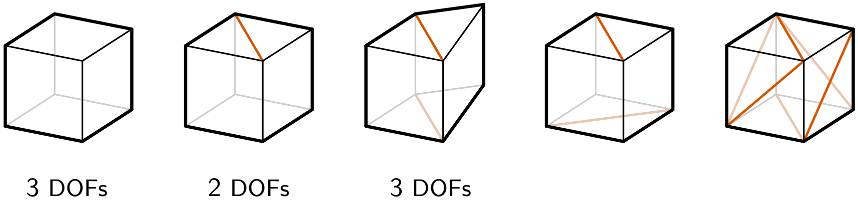

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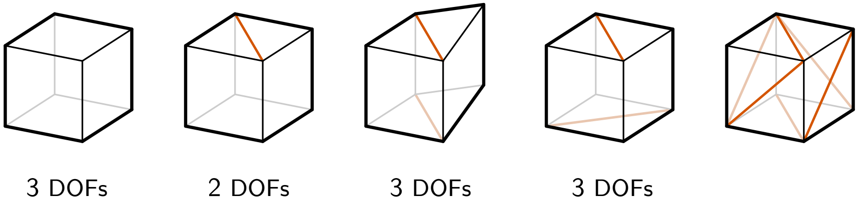

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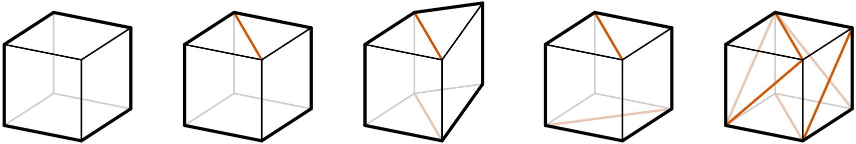

3 DOFs 2 DOFs 3 DOFs 3 DOFs 0 DOFs

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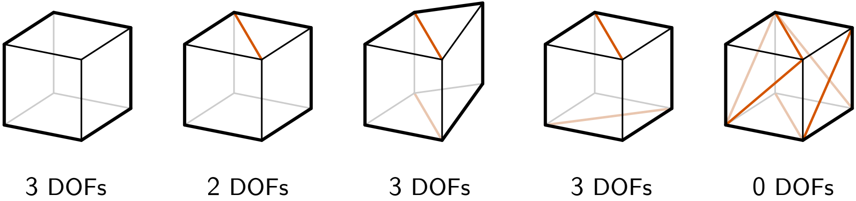
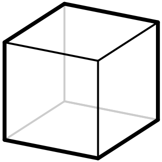
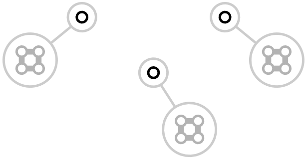


Conclusion:

- too unpredictable
- not strictly convex

Stacking … is bad too, but at least manageable





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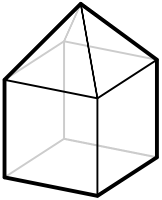

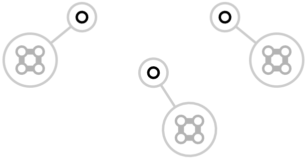

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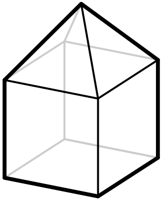

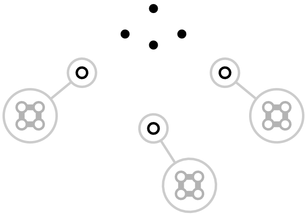

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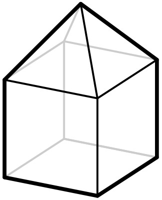

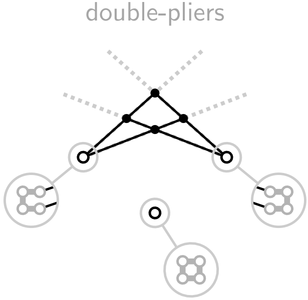

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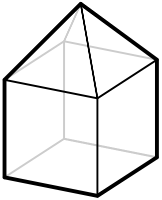

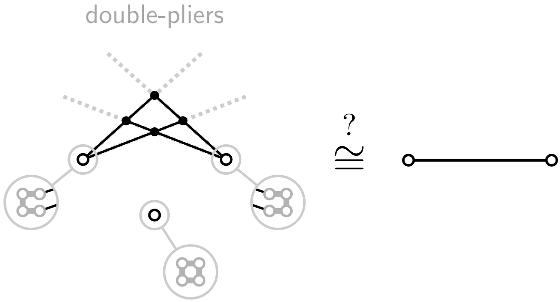

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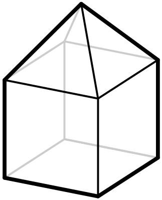
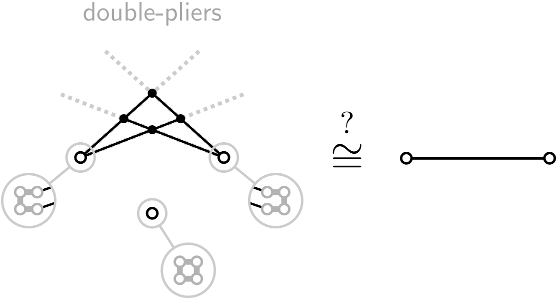
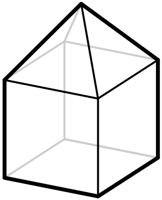




Question: do double-pliers simulate bar constraints? Do they enforce that

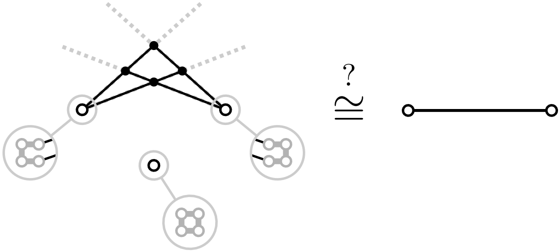
- clusters stay together?
- clusters stay at constant distance?

STACKING ... IS BAD TOO, BUT AT LEAST MANAGEABLE



2 DOFs

double-pliers



Question: do double-pliers simulate bar constraints? Do they enforce that

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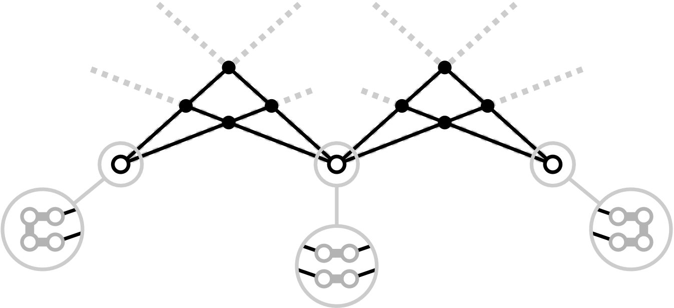
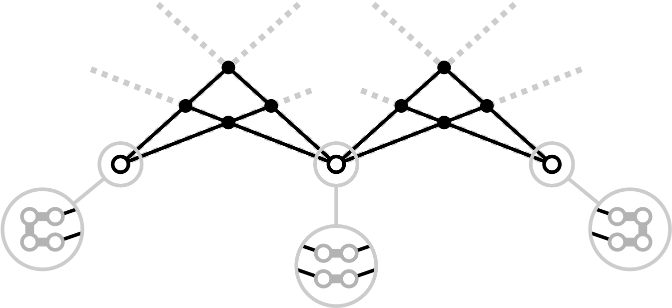
Degree two



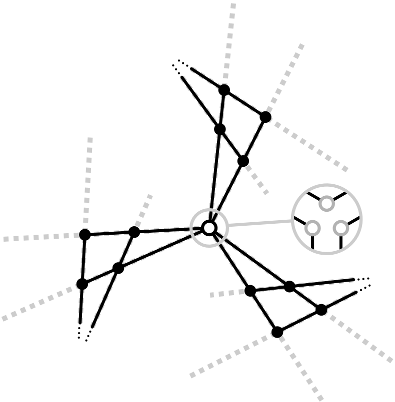
Degree two



Very technical lemma

Clusters at degree two vertices stay together.

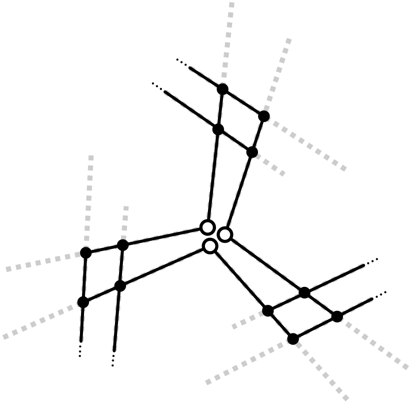
Degree three



Observations:

- Clusters at degree-3 vertices can disintegrate. (1 DOFs)

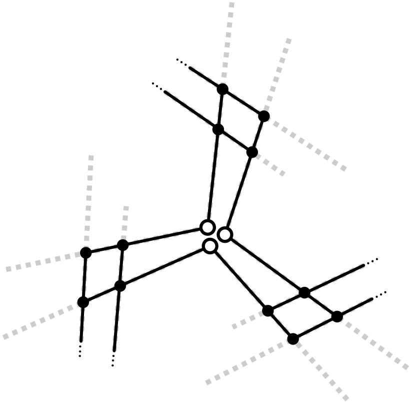
Degree three



Observations:

- Clusters at degree-3 vertices can disintegrate. (1 DOFs)

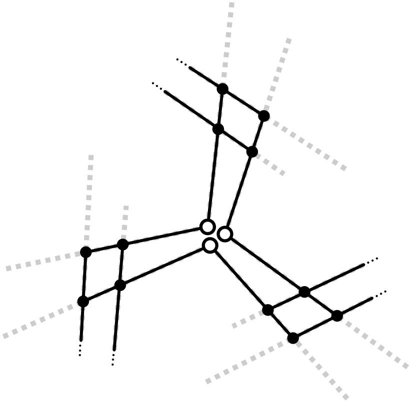
Degree three



Observations: $(k \geq 3)$

- Clusters at degree-k vertices can disintegrate. $(k-2$ DOFs)

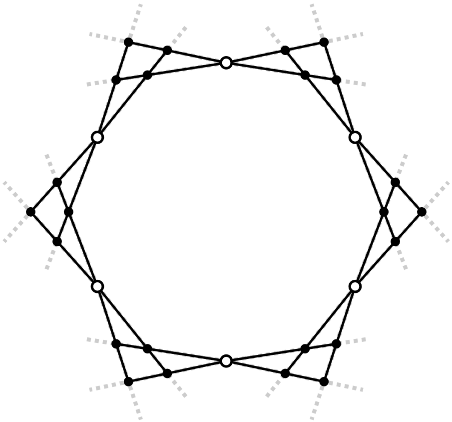
Degree three



Observations: $(k \geq 3)$

- Clusters at degree-k vertices can disintegrate. ($k-2$ DOFs)
- If one cluster of some double-pliers stays together, then so does the other.
  $\longrightarrow$ In a connected graph, it suffices if a single cluster stays together

Many more technical challenges



- Cycles impose constraints on the disintegration of clusters
- Generic choice of pyramid apexes makes these constraints unattainable.

Thank you.

- Polytope rigidity studies deformations of polytopes that preserve edge lengths and coplanarities
- Almost all realizations of a 3-polytope are rigid (true for $d \geq 4$?)
- Concrete cases are hard to decide (is the *truncated dodecahedron* rigid?)
- In general, local rigidity behavior of polytopes can be universally complicated

I. *"Rigidity of polytopes with edge length and coplanarity constraints"*
with Matthias Himmelmann and Bernd Schulze; arXiv:2505.00874

II. *"Second-order rigidity and deformations of polytopes with edge length and coplanarity constraints"*
with Matthias Himmelmann and Zhen "Albert" Zhang (coming soon)

III. *"Higher-dimensional grid bracing and universality of polytope rigidity"*
with Bernd Schulze (coming some day)