# Being non-afraid of non-convexity

## – A minimalist framework for geometric PL complexes –

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August 7, 2026

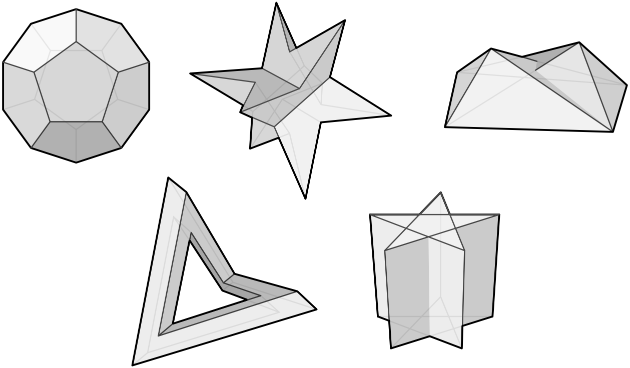
combinatorial synergies

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

# The language of non-convexity

*"Are your Polyhedra the same as my Polyhedra?"*

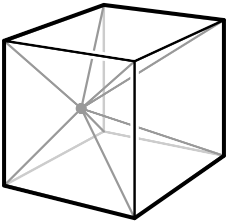
– Branko Grünbaum



# Motivation

- dual volumes and the canonical form
- existence of the Wachspress stress
- strong stress-flex conjecture
- spatial Maxwell-Cremona correspondence

# Rigidity of coned polytope frameworks



**Theorem.** (Pachyli, Prosanov, W., 2026)

*Coned polytope frameworks are prestress stable (in particular, second-order rigid).*

**Ingredients:**

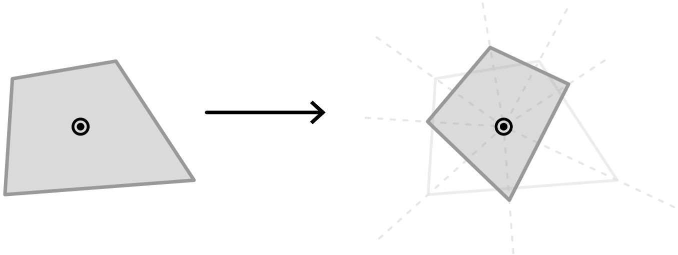
- choose a special stress $\boldsymbol{\omega} : E \to \mathbb{R}$:

$$\sum_{j:ij\in E} \omega_{ij}(p_j - p_i) = 0 \qquad \text{for all } i \in V$$

- show that it satisfies the stress-flex conjecture

# The Wachspress stress

$$P^\circ = \left\{ x \in \mathbb{R}^d \mid \langle x, y \rangle \le 1 \text{ for all } y \in P \right\}.$$



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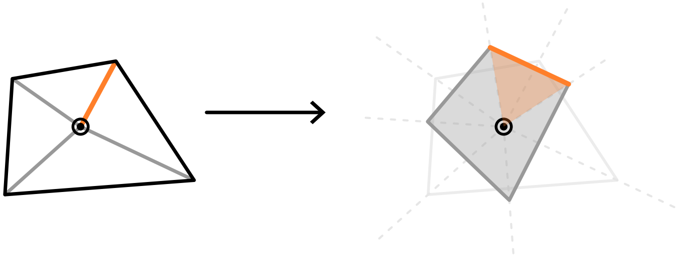

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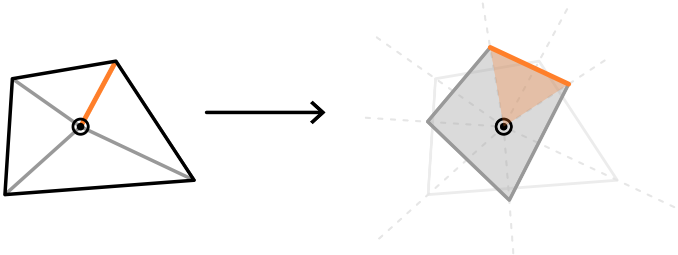


One can show that this is a stress using **Minkowski balancing**:

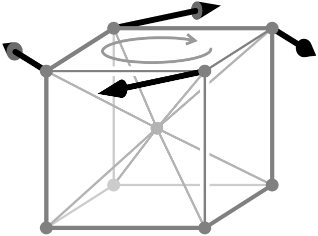
$$\sum_{\sigma \text{ codim } 1} V_\sigma n_\sigma = 0$$

# Stress flex conjecture

**Conjecture.**

*Given a coned polytope framework with infinitesimal flex $\dot{\boldsymbol{p}}$ and stress $\boldsymbol{\omega}$, then*

$$\sum_{i \neq \star} \omega_{i\star} \dot{p}_i = \dot{p}_\star.$$



# Generalized Schläfli formulas

Given a differentiable 1-parameter family of polyhedra $P(t)$, it holds

$$\sum_e \dot{\theta}_e \operatorname{len}(e) \quad = \quad 0$$

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# Generalized Schläfli formulas

Given a differentiable 1-parameter family of polyhedra $P(t)$, it holds

$$\sum_{\tau \text{ codim } 2} \dot{\theta}_\tau \operatorname{vol}(\tau)\, b_\tau = \sum_{\sigma \text{ codim } 1} \dot{V}_\sigma n_\sigma$$

(Souam, Schlenker)

This is dual to the **stress-flex conjecture** for the **Wachspress stress**.

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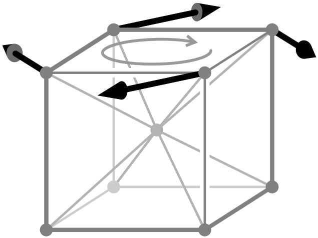

# Stress-flex conjecture

**Conjecture.**

*Given a coned PL surface framework with infinitesimal flex $\dot{\boldsymbol{p}}$ and stress $\boldsymbol{\omega}$, then*

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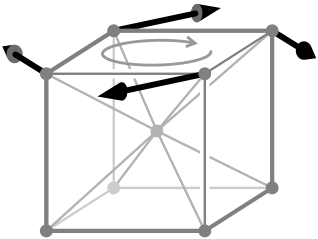
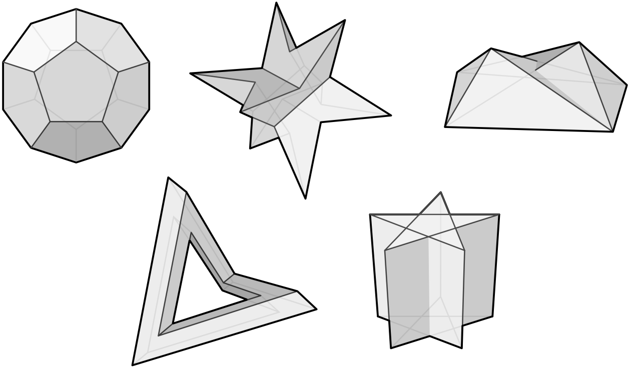


# How much non-convex do we want?

- higher genus
- self-intersecting
- faces of general combinatorics/geometry
- ...

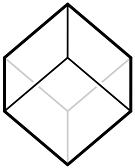


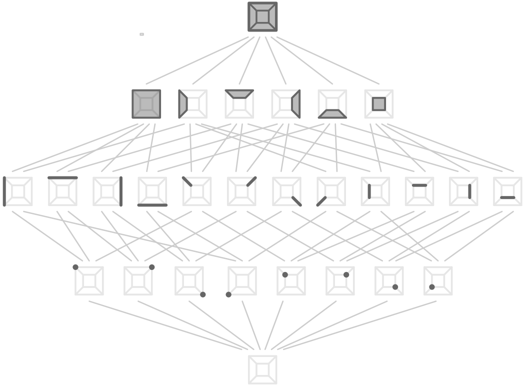
**Key insight:** all we need is to impose certain **coplanarities**.

# The combinatorial structure

# Face lattices

= a **bounded graded lattice** capturing the combinatorics of a polytope.





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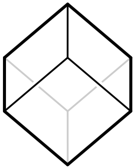
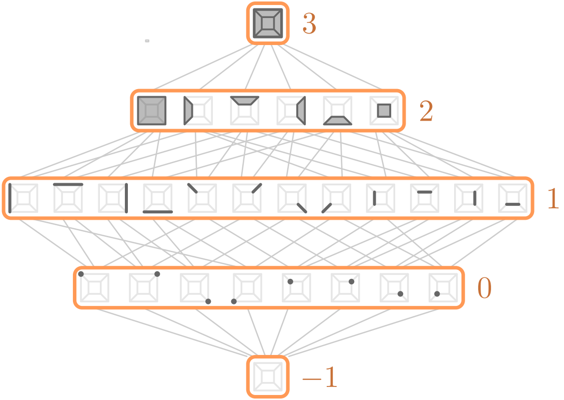


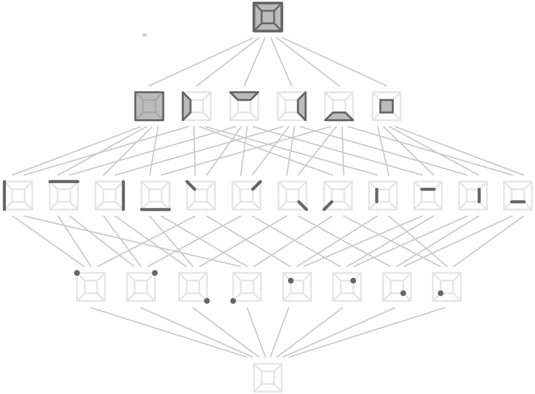


# ABSTRACT POLYTOPES (OR BETTER, ABSTRACT PL MANIFOLDS)

An **abstract polytope** is a bounded graded lattice that satisfies:

(i) diamond property ($\simeq$ manifoldness)

(ii) strongly flag connected (= the flag graph of every interval is connected)



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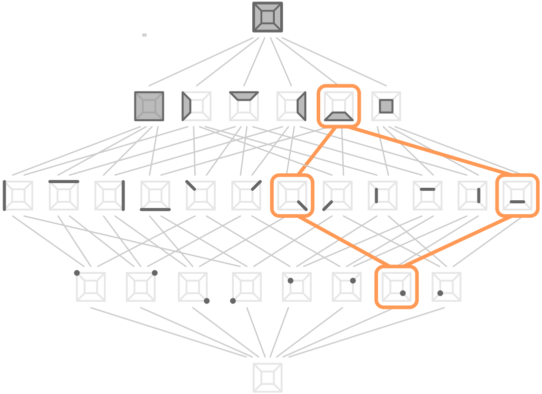

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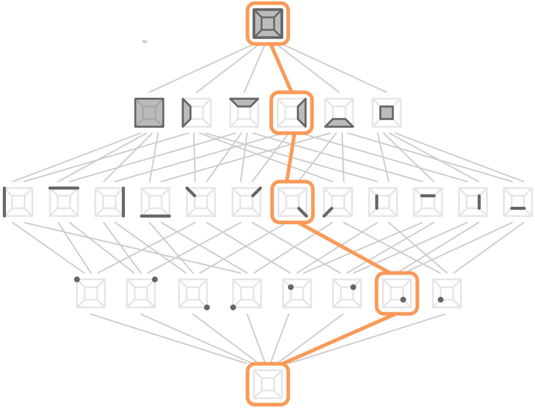

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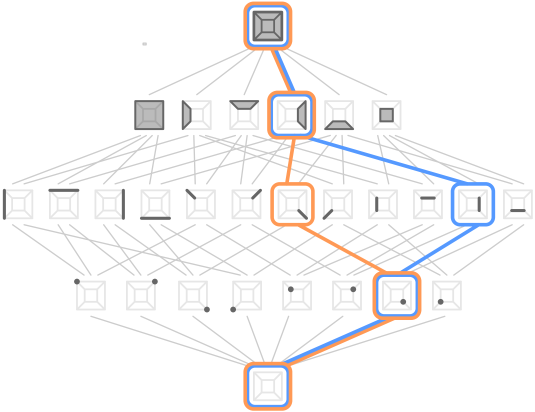

# The geometric realization

# Realizations

An **affine realization** of $\mathcal{P}$ is a map

$$\sigma \mapsto \mathrm{aff}(\sigma),$$

that to every face $\sigma \in \mathcal{P}$ assigns an affine subspace $\mathrm{aff}(\sigma)$, so that

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That's it. There is no mention of ...

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- boundary,
- polytopes, polygons, line segments or other flat or convex pieces,
- gluing conditions,
- non-degeneracy or non-coincidence conditions

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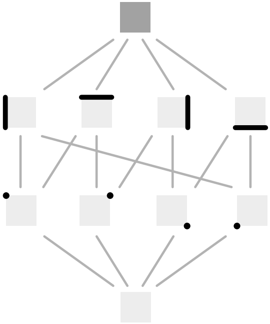
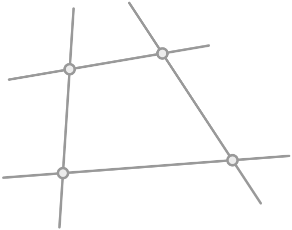
**Claim:** This (almost) suffices for all we want to do.

# Example





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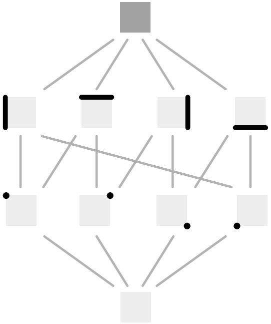

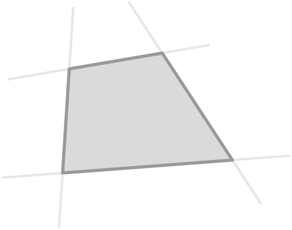

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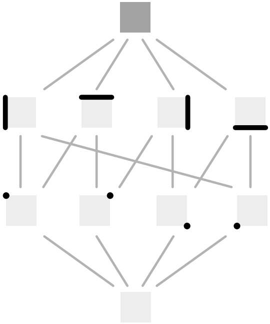 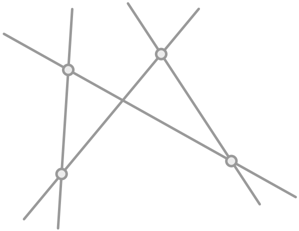

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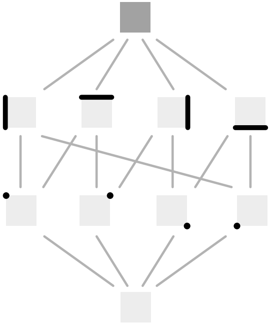
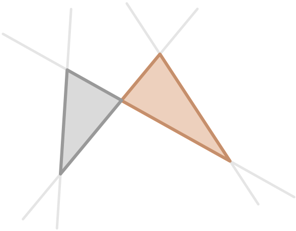
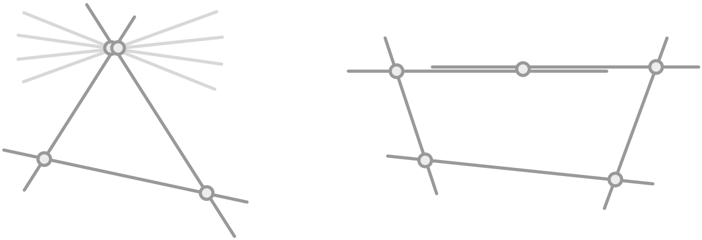


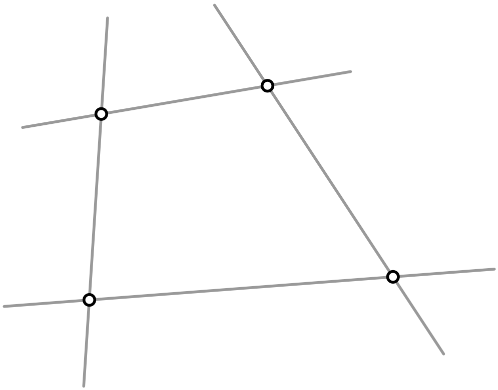


# Example (degeneracy is explicitly allowed)



# Volume, polarity and other things

# Computing volume



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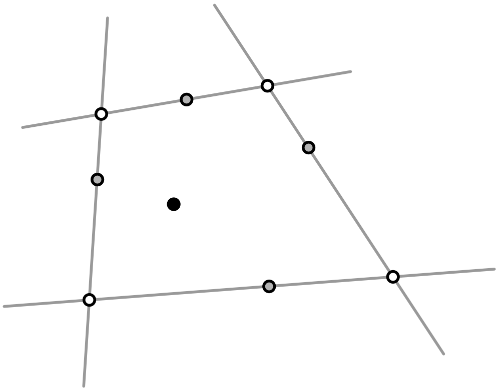

- A **pointing** maps $\sigma \mapsto x_\sigma \in \operatorname{aff}(\sigma)$.

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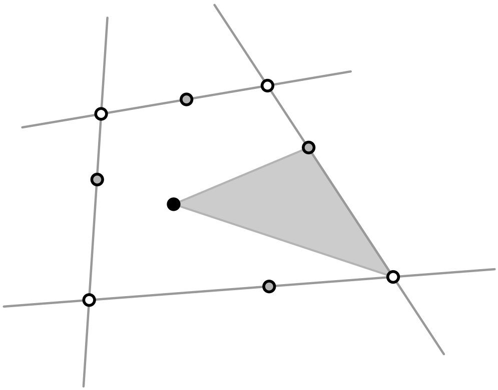

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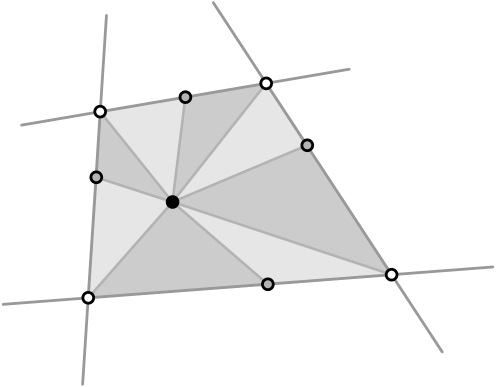

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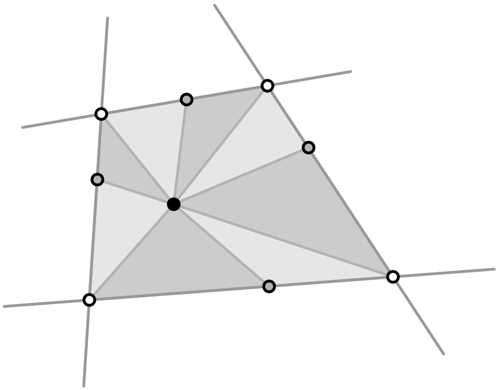

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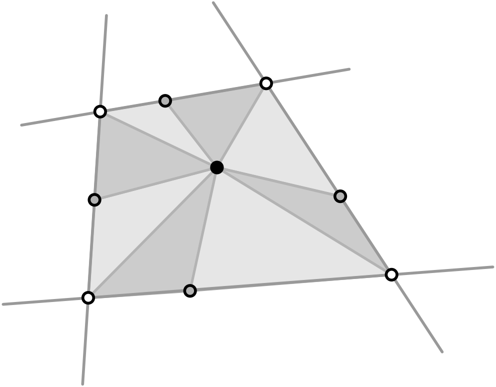

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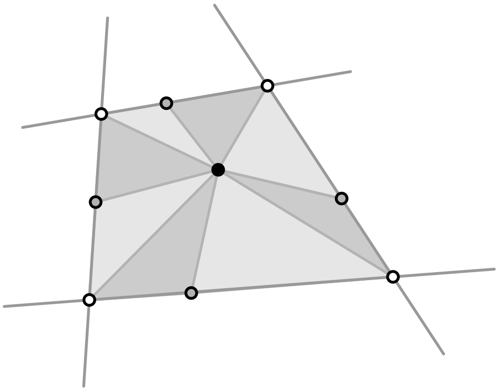
# Defining volume



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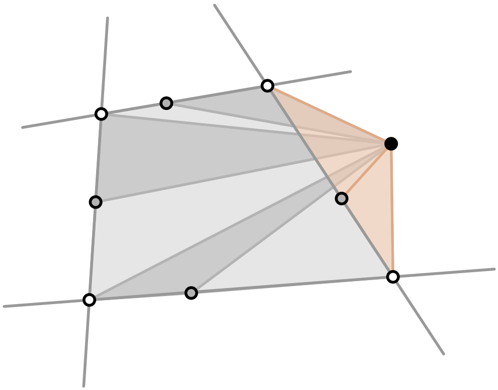

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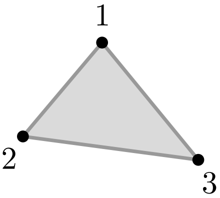
# Orientation

- $\mathcal{P}$ is **orientable** if the flag graph is bipartite.
- an **combinatorial orientation** of $\mathcal{P}$ is bipartition of the flag graph into classes $+1$ and $-1$.
- A **geometric orientation** of $\mathbb{R}^d$ is selection of signs for $d$ affinely independent points.



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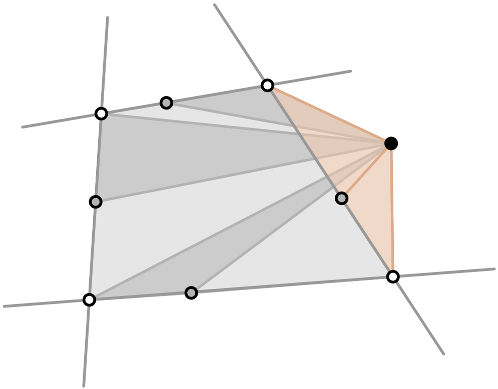

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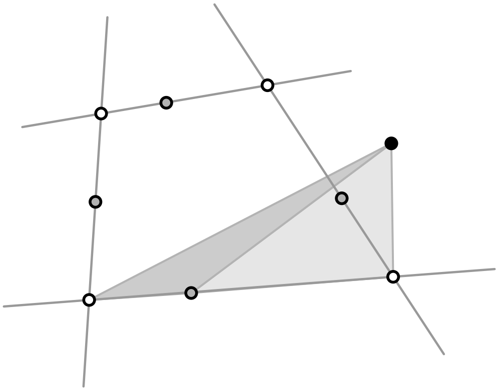

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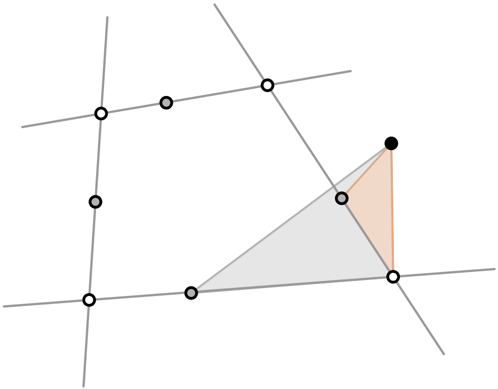

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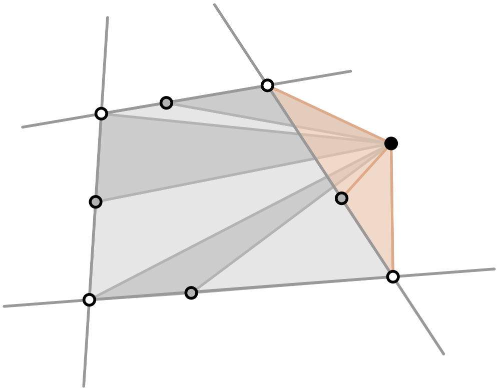

# Defining volume with orientation in mind

$$\mathrm{vol}(\mathcal{P}, \mathrm{aff}) := \sum_{\boldsymbol{\sigma}} \mathrm{sgn}(\boldsymbol{\sigma})\, \mathrm{sgn}(\Delta_{\boldsymbol{\sigma}})\, \mathrm{vol}(\Delta_{\boldsymbol{\sigma}}).$$

# Defining volume with orientation in mind

A **valuation** is a map $\phi : \mathfrak{P}^d \to A$ with

$$\phi(P) + \phi(Q) = \phi(P \cap Q) + \phi(P \cup Q)$$

whenever $P, Q, P \cup Q, P \cap Q$ are convex polytope.

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$$\phi(\mathcal{P}, \mathrm{aff}) := \sum_{\boldsymbol{\sigma}} \mathrm{sgn}(\boldsymbol{\sigma})\, \mathrm{sgn}(\Delta_{\boldsymbol{\sigma}}) \phi(\Delta_{\boldsymbol{\sigma}}) \quad .$$

A valuation is **simple** if $\phi(P) = 0$ whenever $\dim(P) < d$. Equivalently

$$\phi(P_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_n) = \phi(P_1) + \cdots + \phi(P_n).$$

**Theorem.** (M., 2026+)

*If $\phi$ is simple, then $\phi(\mathcal{P}, \mathrm{aff})$ does not depend on the pointing.*

# Many things are simple valuation

Simple valuations:

- volume
- odd surface area measure
- Dehn invariant $\sum_e \ell_e \otimes_{\mathbb{Z}} (\theta_e/\pi)$
- dual volume $\mathrm{vol}(P-x)^\circ$ (canonical form)
- indicator functions modulo lower dimensional polytopes
- normal vectors weighted by facet volume (on a fixed hyperplane)
- dihedral angles modulo $\pi$ (on a fixed codim-2 subspace)

Valuative identities:

- Minkowski balancing
- Schläfli formula (and its extensions)
- stress-flex orthogonality

# Going beyond simple valuations

- Not every valuation extends to this setting
- Non-simple valuations require geometric orientations on the affine subspaces
- Some valuations do not extend uniquely
- It works for general *isometry-covariant valuations*
  e.g. intrinsic volumes

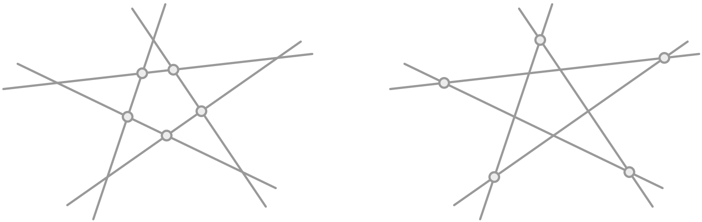
# Thank you.

- Many results about PL manifolds do not rely on being convex, genus zero, being embedded, "nice" faces, ...
- Realizations of abstract polytopes via affine subspaces suffices to define a wide range of natural geometric quantities (including simple valuations) and identities.

*"Second-order rigidity of coned polytope frameworks and the stress-flex conjecture from a vector-valued Schläfli formula"*.

with Eleni Pachyli and Roman Prosanov. arXiv:2607.14878

# Example



# Proof of independence

**Idea:** Lift the valuation to a universal simple valuation, and prove it there.

- Let $A$ be the abelian group generated by formal oriented $d$-simplices $[x_0, ..., x_d]$, factored by relations:
  - $[x_{s(0)}, ..., x_{s(d)}] = \mathrm{sgn}(s)[x_0, ..., x_d]$
  - $[x_0, ..., x_d] = 0$ if the points are coplanar
  - $[\Delta] = [\Delta_1] + \cdots + [\Delta_n]$ if the $\Delta_i$ tile $\Delta$ and have the same orientations.
- The map $\Delta \mapsto [\Delta]$ extends to a simple valuation $i : \mathfrak{P}^d \to A$.
- Every simple valuation factors through $i$: $\phi(P) = \tilde{\phi}(i(P))$.
- It suffices to prove that $i(\mathcal{P}, \mathrm{aff})$ is independent of the pointing.
- *[technical argument about simplicial chains using the diamond property]*

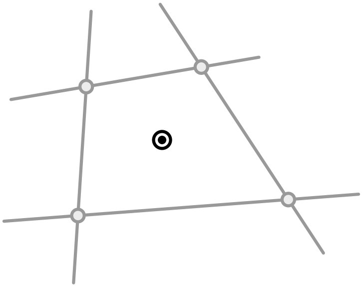
# Polar duality

**Combinatorially:**

- $\dim(\sigma^\diamond) = d - \dim(\sigma^\diamond) - 1$

**Geometrically:**

- $\operatorname{aff}(\sigma^\diamond) := \operatorname{aff}(\sigma)^\perp$
- $\operatorname{dist}(\operatorname{aff}(\sigma^\diamond)) = \operatorname{dist}(\operatorname{aff}(\sigma))^{-1}$



# Polar duality

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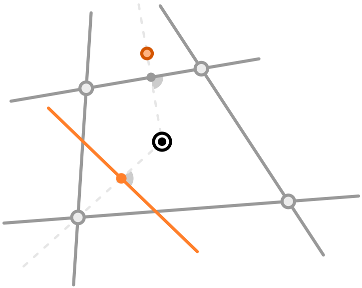
- $\dim(\sigma^\diamond) = d - \dim(\sigma^\diamond) - 1$

**Geometrically:**

- $\mathrm{aff}(\sigma^\diamond) := \mathrm{aff}(\sigma)^\perp$
- $\mathrm{dist}(\mathrm{aff}(\sigma^\diamond)) = \mathrm{dist}(\mathrm{aff}(\sigma))^{-1}$



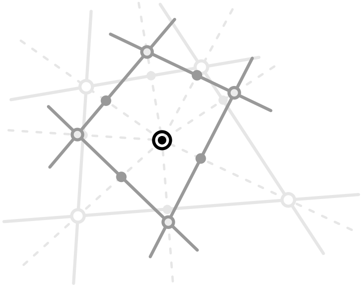
# Polar duality

**Combinatorially:**

- $\dim(\sigma^\diamond) = d - \dim(\sigma^\diamond) - 1$

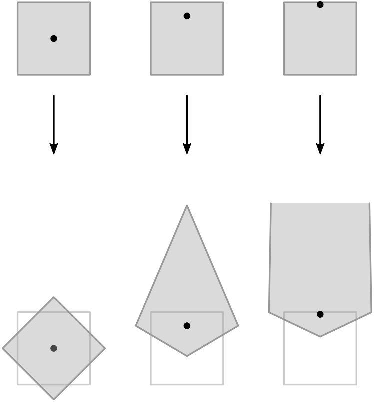
**Geometrically:**

- $\operatorname{aff}(\sigma^\diamond) := \operatorname{aff}(\sigma)^\perp$
- $\operatorname{dist}(\operatorname{aff}(\sigma^\diamond)) = \operatorname{dist}(\operatorname{aff}(\sigma))^{-1}$



# Polarity beyond convexity

$$P^\circ = \big\{ x \in \mathbb{R}^d \mid \langle x, y\rangle \le 1 \text{ for all } y \in P \big\}.$$



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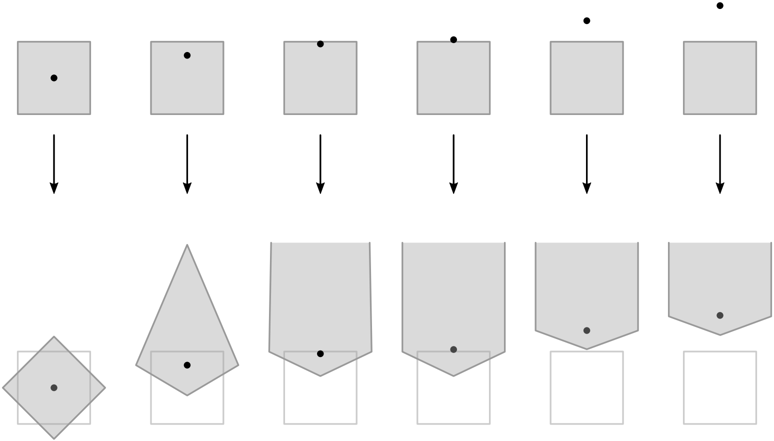
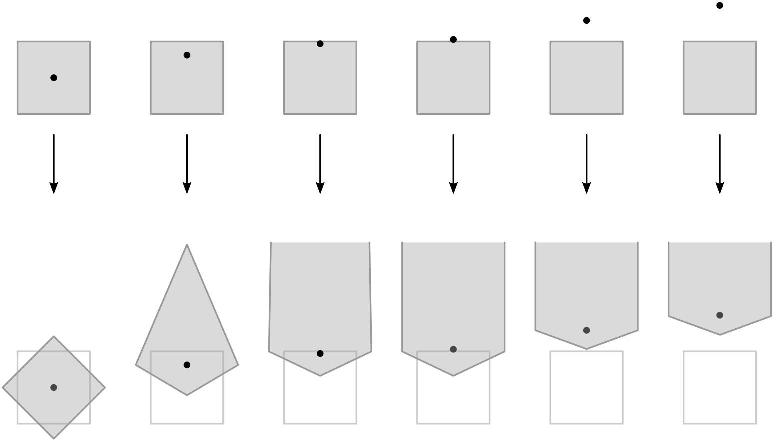


# Polarity beyond convexity

$$\operatorname{vol}(P-x)^\circ = \frac{p(x)}{q(x)}$$



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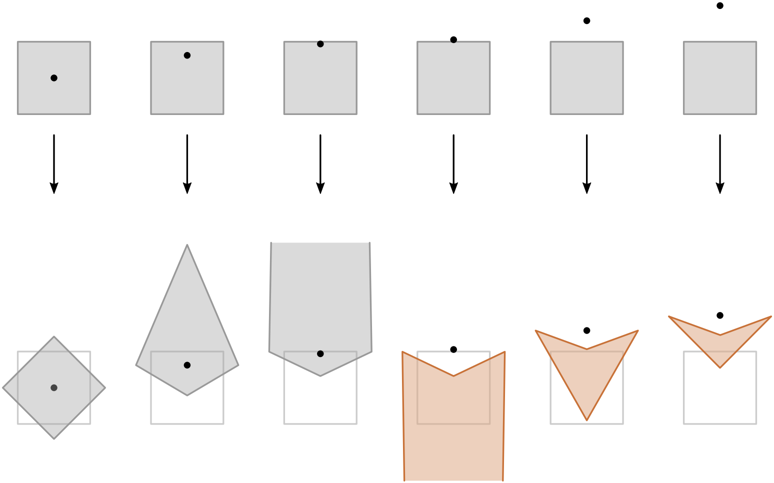
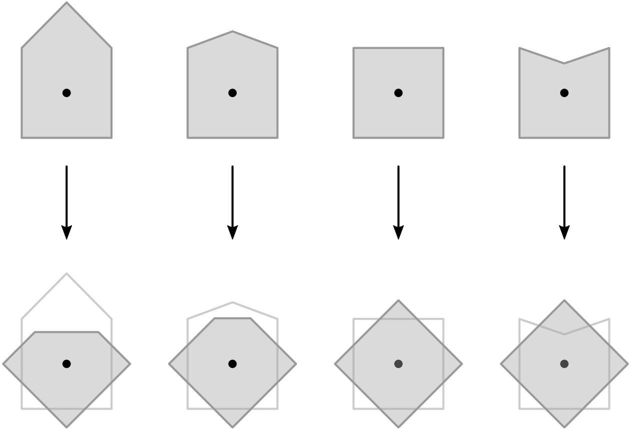
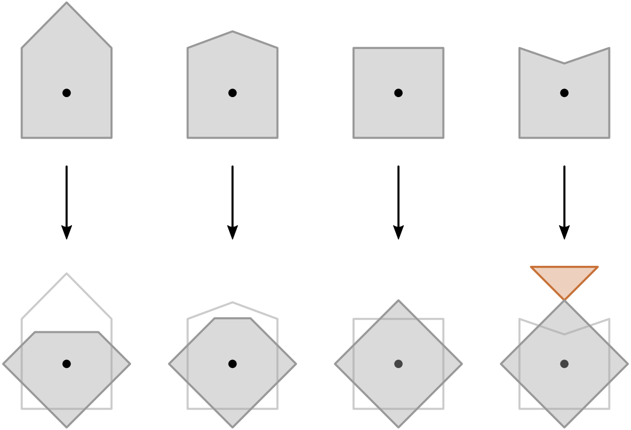


# Polarity beyond convexity



# Polarity beyond convexity



# Polarity beyond convexity